Вестник ЧГПУ им. И. Я. Яковлева
Серия: Механика предельного состояния. 2009. № 1. С. 179–194

N. Takeuchi, A. Vardanyan

ELASTO-PLASTIC ANALYSIS FOR PLATE BENDING PROBLEMS BY USING HYBRID-TYPE PENALTY METHOD

Department of Art and Technology, Hosei University,

Institute of Mechanics, Armenian National Academy of Sciences

Abstract. In present paper, we have given the investigations of the plate bending problem by numerical treatment using hybrid-type penalty method (HPM). The HPM assume linear and non-linear displacement field with rigid displacement, rigid rotation, strain and its gradient in each subdomain and introduce subsidiary condition about the continuity of displacement into the framework of the variational expression with Lagrange multipliers. For the purpose of this paper, we accepted the Kirchhoff theory that neglects the transversal shear deformation. In the first step of the work, we are giving the equilibrium equations for a deformable body in 3D case and as boundary conditions we are giving geometrical (for displacement field) and kinetic (for surface force) boundary conditions. Secondary we apply Kirchhoff theory to the displacement field of the 3D case for plate bending problem. For this purpose, we use quadratic form that includes rigid, linear and nonlinear parts of the displacements. The parameters used in this displacement field are independently defined in each subdomain. We introduce penalty function that presents strong spring connecting each subdomain. Then, we take the matrix of the subsidiary condition according to the surface integral of the contact surface of each sub-domain. We apply nonlinearity in penalty function such as spring system, which allow us to calculate hinge line. If hinge line makes mechanism then we can calculate limit load. We used load incremental method called r-min method in a material nonlinear analysis. We can calculate growing hinge line systematically using this algorithm for the nonlinear analysis. Finally, we calculate some simple problems to check accuracy of elastic solution and limit load.

Keywords: plate bending, penalty method, hybrid-type virtual work, discontinuous Galerkin method

УДК: 539.375

1. Introduction.

In this work for obtaining numerical results for the plate bending problem, we have used hybrid-type penalty method (HPM), which applied the concept of the penalty method [1] to the principle of hybrid type virtual work [7]. The HPM is based on discontinuous Galerkin (dG) method [4]. The HPM applies the concept of the spring of RBSM [3] (Rigid Bodies-Spring Model) in Lagrange multiplier and assume independent displacement field to each subdomain. Because compatibility requirements of the intersection boundary on adjacent sub-domain are secured by using the penalty method, the displacement field can be assumed regardless of the shape of sub-domain [6]. However, we cannot obtain high

Поступила 11.03.2008

accuracy solutions when it uses shape other than the triangle at the linear displacement field. For the reason, it is difficult to use the mesh division of arbitrary shape. To solve such a problem, we proposed the method of applying the second-order displacement field that added the gradient of the strain to linear displacement field of HPM [5].

Usually, for the finite element method, it requires C^1 continuity. However, many plate elements and numerical results are not satisfying this completely. Then, by using C^0 elements instead, it imposes the continuity of slope weakly. It calls this method discontinuous Galerkin method [2].

In this paper, we proposed the new element model on the plate bending problems by using HPM based on the dG method. In the first part of the paper, we have given brief formulation of proposed method, and in second part, we have given some numerical results.

2. Governing equation and hybrid-type virtual work

2.1. Governing equation. Let $\Omega \subset \mathrm{R}^{n_{\dim}}$, with $(1 \leq n_{\dim} \leq 3)$, be the reference configuration of a continuum body with smooth boundary $\Gamma := \partial\Omega$ and closure $\bar{\Omega} := \Omega \cup \partial\Omega$. Here $\mathrm{R}^{n_{\dim}}$ is the n_{dim} dimensional Euclidean space.

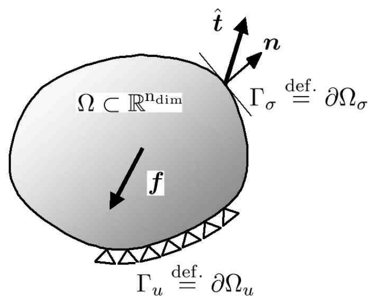

Fig. 1. Reference configuration Ω and smooth boundary $\partial\Omega$

The local form of the equilibrium equation for a deformable body is as follows:

$$\operatorname{div} \sigma + \mathbf{f} = 0 \quad in \quad \Omega, \tag{1}$$

$$\sigma = \sigma^t \quad in \quad \Omega, \tag{2}$$

where, $\mathbf{f} : \Omega \to \mathrm{R}^{n_{\dim}}$ is the body force per unit volume, $\sigma : \bar{\Omega} \to \mathrm{S}$ is the Cauchy stress tensor respectively. Here, $\mathrm{S} = \mathrm{R}^{(n_{\dim}+1)\cdot n_{\dim}/2}$ is the vector space of symmetric rank-two tensor and $\mathbf{e}_i$ is the standard base vector of $\mathrm{R}^{n_{\dim}}$, so that the stress tensor becomes $\sigma = \sigma_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, where $\otimes$ denotes a tensor product. $\mathbf{u} : \bar{\Omega} \to \mathrm{R}^{n_{\dim}}$ is a displacement field of particles with reference position $\mathbf{x} \in \Omega$. We write this displacement field to be $\mathbf{u}(\mathbf{x})$ and denote the infinitesimal strain tensor by

$$\varepsilon = \nabla^s \mathbf{u} := \frac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^t\right], \tag{3}$$

where $\nabla := (\partial/\partial x_i)\,\mathbf{e}_i$ is the differential vector operator, ∇^s shows the symmetry part of ∇.

Then, we assume that the boundary $\Gamma := \Gamma_u \cup \Gamma_\sigma$.

$$\Gamma = \overline{\Gamma_u \cup \Gamma_\sigma}, \;\; \Gamma_u \cap \Gamma_\sigma = \emptyset \tag{4}$$

here $\Gamma_u := \partial_u \Omega \subset \partial\Omega$ where displacement are prescribed as

$$\mathbf{u}|_{\Gamma_u} = \hat{\mathbf{u}} \ \ (given) \tag{5}$$

where as $\Gamma_\sigma := \partial_\sigma \Omega \subset \partial\Omega$ where tractions $\mathbf{t} := \sigma\mathbf{n}$ are prescribed as

$$\sigma|_{\Gamma_\sigma} \hat{\mathbf{n}} = \hat{\mathbf{t}} \ \ (given). \tag{6}$$

Here $\hat{\mathbf{n}}$ is the field normal to the boundary Γ_σ. The constitutive equation to the elastic body is provided as follows by using the elasticity tensor $\mathbf{C}$.

$$\sigma = \mathbf{C} : \varepsilon. \tag{7}$$

2.2. Virtual work equation (weak forms). Let denote by $\boldsymbol{U}$ the space of admissible displacement field, defined as

$$\mathrm{U} :=? \left\{ \mathbf{u} : \Omega \to \mathrm{R}^{n_{\dim}} | \mathbf{u}|_{\Gamma_u} = \hat{\mathbf{u}} \right\}. \tag{8}$$

And, let denote by $\boldsymbol{V}$ the space of admissible virtual displacement field, defined as

$$\mathrm{V} := \left\{ \delta\mathbf{u} : \Omega \to \mathrm{R}^{n_{\dim}} | \delta\mathbf{u}|_{\Gamma_u} = 0 \right\}. \tag{9}$$

We now use equation (1) and integrate volume of the body to give a weak form of the static equilibrium of the body as

$$\delta W := \int_\Omega (div\,\sigma + \mathbf{f}) \cdot \delta\mathbf{u}\, dV = 0 \quad \forall \delta\mathbf{u} \in \mathrm{V}. \tag{10}$$

A more common and useful expression can be derived to give the divergence of the vector $\sigma\delta\mathbf{u}$ as

$$div\,(\sigma\delta\mathbf{u}) = (div\,\sigma) \cdot \delta\mathbf{u} + \sigma : \mathrm{grad}\,\delta\mathbf{u}. \tag{11}$$

Using this equation together with the Gauss theorem enable equation (10) to be rewritten as

$$\int_\Omega \sigma : \mathrm{grad}\,\delta\mathbf{u}\, dV - \int_\Omega \mathbf{f} \cdot \delta\mathbf{u}\, dV - \int_{\Gamma_\sigma} \hat{\mathbf{t}} \cdot \delta\mathbf{u}\, dS = 0 \quad \forall \delta\mathbf{u} \in \mathrm{V}. \tag{12}$$

This equation is virtual work equation. If $\mathbf{u}$is the weighing function, this is a weak forms. It is $\mathrm{U} \subset \mathrm{H}^1(\Omega)$ and $\mathrm{V} \subset \mathrm{H}^1(\Omega)$ where denotes the Sobolev space $\mathrm{H}^1(\Omega)$ of function possessing space integrable derivatives.

2.3. Hybrid-type virtual work equation. Let Ω consist of M sub-domains $\Omega^{(e)} \subset \Omega$ with the closed boundary $\Gamma^{(e)} := \partial\Omega^{(e)}$ as shown in Figure 2.

That is

$$\Omega = \bigcup_{e=1}^{M} \Omega^{(e)} \quad here \quad \Omega^{(r)} \cap \Omega^{(q)} = 0 \ \ (r \neq q). \tag{13}$$

In what follows, we assume that the closure $\bar{\Omega}^{(e)} := \Omega^{(e)} \cup \partial\Omega^{(e)}$.

We denoted by $\Gamma_{<ab>}$ the common boundary for two sub-domain $\Omega^{(a)}$ and $\Omega^{(b)}$ adjoined as shown in Figure 3, and which is defining as

$$\Gamma_{<ab>} := \Gamma^{(a)} \cap \Gamma^{(b)}. \tag{14}$$

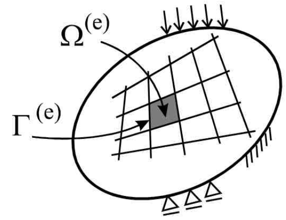

Fig. 2. Sub-domain $\Omega^{(e)}$

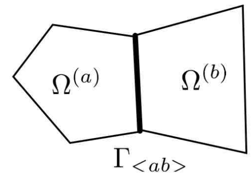

Fig. 3. Common boundary $\Gamma_{<ab>}$ of sub-domain $\Omega^{(a)}$ and $\Omega^{(b)}$

The relation for $\tilde{\mathbf{u}}^{(a)}$ and $\tilde{\mathbf{u}}^{(b)}$ are following:

$$\tilde{\mathbf{u}}^{(a)} = \tilde{\mathbf{u}}^{(b)} \ \ on \ \ \Gamma_{<ab>}. \tag{15}$$

They are the displacements on the intersection boundary $\Gamma_{<ab>}$ in sub-domain $\Omega^{(a)}$ and $\Omega^{(b)}$.

This subsidiary condition is introduced into the framework of the variational equation (12) with Lagrange multipliers λ as follows:

$$H_{<ab>} := \delta \int_{\Gamma_{<ab>}} \lambda \cdot \left(\tilde{\mathbf{u}}^{(a)} - \tilde{\mathbf{u}}^{(b)}\right) dS, \tag{16}$$

where $\delta\,(\bullet)$ shows the variation of $(\bullet)$ Physical meaning of the Lagrange multiplier λ is equal to the surface force on the intersection boundary $\Gamma_{<ab>}$.

$$\lambda = \mathbf{t}^{(a)}(\tilde{\mathbf{u}}^{(a)}) = -\mathbf{t}^{(b)}(\tilde{\mathbf{u}}^{(b)}), \tag{17}$$

where $\mathbf{t}^{(a)}$ and $\mathbf{t}^{(b)}$ are the surface force on the intersection boundary in sub-domain $\Omega^{(a)}$ and $\Omega^{(b)}$. The hybrid type virtual work equation can be described as follows about M subdomain and N intersection boundary:

$$\begin{gathered}\sum_{e=1}^{M}\left(\int_{\Omega^{(e)}} \sigma : \mathrm{grad}\,(\delta\mathbf{u}) dV - \int_{\Omega^{(e)}} \mathbf{f}\cdot\delta\mathbf{u} dV\right) - \sum_{s=1}^{N}\left(\delta\int_{\Gamma_{<s>}} \lambda(\tilde{\mathbf{u}}^{(a)} - \tilde{\mathbf{u}}^{(b)}) dS\right) - \\ - \int_{\Gamma_\sigma} \hat{\mathbf{t}}\cdot\delta\mathbf{u} dS = 0 \ \ \forall\delta\mathbf{u}\in \mathrm{V}.\end{gathered} \tag{18}$$

3. Independent displacement field and relative displacement

3.1 3D displacement fields. In the following work, we consider three-dimensional displaced field $\mathbf{u} \in \mathrm{U}$ with $\mathrm{n}_{dim} = 3$ and carry out Taylor's expansion of displacement $\mathbf{u}(\mathbf{x})$ for point $\mathbf{x}_p = (x_p, y_p, z_p) \in \Omega^{(e)}$ from $\Omega^{(e)}$ arbitrary domain. Consequently, the second-order displacement field in the arbitrary sub-domain $\Omega^{(e)}$ is as follows by matrix form:

$$\mathbf{u}^{(e)} = \mathbf{N}_d^{(e)}\mathbf{d}^{(e)} + \mathbf{N}_\varepsilon^{(e)}\varepsilon^{(e)} + \mathbf{N}_{gx}^{(e)}\varepsilon_{,x}^{(e)} + \mathbf{N}_{gy}^{(e)}\varepsilon_{,y}^{(e)} + \mathbf{N}_{gz}^{(e)}\varepsilon_{,z}^{(e)}, \tag{19}$$

where

$$\mathbf{N}_d^{(e)} = \begin{bmatrix} 1 & 0 & 0 & 0 & Z & -Y \\ 0 & 1 & 0 & -Z & 0 & X \\ 0 & 0 & 1 & Y & -X & 0 \end{bmatrix}, \quad \mathbf{N}_\varepsilon^{(e)} = \begin{bmatrix} X & 0 & 0 & Y/2 & 0 & Z/2 \\ 0 & Y & 0 & X/2 & Z/2 & 0 \\ 0 & 0 & Z & 0 & Y/2 & X/2 \end{bmatrix},$$

$$\mathbf{N}_{gx}^{(e)} = \begin{bmatrix} X^2/2 & -Y^2/2 & -Z^2/2 & 0 & -YZ/2 & 0 \\ 0 & XY & 0 & X^2/2 & ZX/2 & 0 \\ 0 & 0 & ZX & 0 & XY/2 & X^2/2 \end{bmatrix},$$

$$\mathbf{N}_{gy}^{(e)} = \begin{bmatrix} XY & 0 & 0 & Y^2/2 & 0 & YZ/2 \\ -X^2/2 & Y^2/2 & -Z^2/2 & 0 & 0 & -ZX/2 \\ 0 & 0 & YZ & 0 & Y^2/2 & XY/2 \end{bmatrix},$$

$$\mathbf{N}_{gz}^{(e)} = \begin{bmatrix} ZX & 0 & 0 & YZ/2 & 0 & Z^2/2 \\ 0 & YZ & 0 & ZX/2 & Z^2/2 & 0 \\ -X^2/2 & -Y^2/2 & Z^2/2 & -XY/2 & 0 & 0 \end{bmatrix},$$

$$\mathbf{d} = [u_p, v_p, w_p, \theta_x, \theta_y, \theta_z]^t, \quad \varepsilon = [\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}]^t,$$

$$\varepsilon_{,x} = [\varepsilon_{x,x}, \varepsilon_{y,x}, \varepsilon_{z,x}, \gamma_{xy,x}, \gamma_{yz,x}, \gamma_{zx,x}]^t,$$

$$\varepsilon_{,y} = [\varepsilon_{x,y}, \varepsilon_{y,y}, \varepsilon_{z,y}, \gamma_{xy,y}, \gamma_{yz,y}, \gamma_{zx,y}]^t,$$

$$\varepsilon_{,z} = [\varepsilon_{x,z}, \varepsilon_{y,z}, \varepsilon_{z,z}, \gamma_{xy,z}, \gamma_{yz,z}, \gamma_{zx,z}]^t,$$

$$X = x - x_p, \quad Y = y - y_p, \quad Z = z - z_p.$$

3.2. Displacement for thin plate. The thin plate is defining as follows:

$$\Omega := \left\{ (x, y, z) \in \mathrm{R}^3 | z \in [-t/2, t/2], (x, y) \in A \in \mathrm{R}^2 \right\}, \tag{20}$$

where is plate thickness and A is plate area.

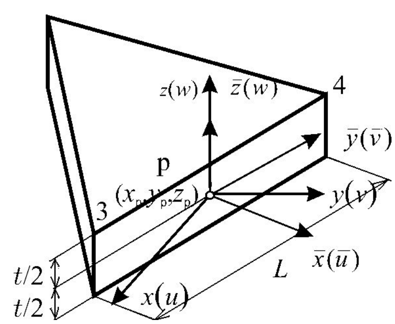

Fig. 4. Thin plate

Since for thin plate we have following Kirchhoff-Love's assumptions:

$$\varepsilon_z = \gamma_{yz} = \gamma_{zx} = 0 \tag{21}$$

we will obtain deflection for thin plate as follows:

$$w = w_p + Y\theta_x^p - X\theta_y^p - \frac{1}{2}X^2\varepsilon_{x,z}^p - \frac{1}{2}Y^2\varepsilon_{y,z}^p - \frac{1}{2}XY\gamma_{xy,z}^p. \tag{22}$$

Consequently, the displacement at arbitrary point will be:

$$\mathbf{u}^{(e)} = \mathbf{Z}_M\mathbf{N}_{Md}^{(e)}\mathbf{d}_M^{(e)} + \mathbf{Z}_M\mathbf{N}_{Mq}^{(e)}\varepsilon_M^{(e)}, \tag{23}$$

where

$$\mathbf{Z}_M = \begin{bmatrix} -z & 0 & 0 \\ 0 & -z & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \mathbf{N}_{Md}^{(e)} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & Y & -X \end{bmatrix},\ \mathbf{N}_{Mq}^{(e)} = \begin{bmatrix} -X & 0 & -\frac{Y}{2} \\ 0 & -Y & -\frac{X}{2} \\ -\frac{X^2}{2} & -\frac{Y^2}{2} & -\frac{XY}{2} \end{bmatrix},$$

$$\mathbf{d}_M^{(e)} = \begin{Bmatrix} w_p \\ \theta_x^p \\ \theta_y^p \end{Bmatrix},\ \varepsilon_M^{(e)} = \begin{Bmatrix} \varepsilon_{x,z}^p \\ \varepsilon_{y,z}^p \\ \varepsilon_{xy,z}^p \end{Bmatrix},\ \mathbf{u}^{(e)} = \begin{Bmatrix} u^{(e)} \\ v^{(e)} \\ w^{(e)} \end{Bmatrix},$$

(23) we can write by matrix form:

$$\mathbf{u}^{(e)} = \mathbf{Z}_M\mathbf{N}^{(e)}\mathbf{U}^{(e)}, \tag{24}$$

where

$$\mathbf{N}^{(e)} = \left\lfloor \mathbf{N}_{Md}^{(e)}\mathbf{N}_{Mq}^{(e)} \right\rfloor,\ \ \mathbf{U}^{(e)} = \left\lfloor \mathbf{d}_M^{(e)}\varepsilon_M^{(e)} \right\rfloor^t.$$

3.3. Relative displacement. Now we have to do transformation from global coordinate system to local. The local coordinate system matrix form is follows:

$$\tilde{\mathbf{u}}^{(e)} = \mathbf{R}^{(e)}\mathbf{u}^{(e)}, \tag{25}$$

where $\mathbf{R}^{(e)}$ is:

$$\mathbf{R}^{(e)} = \begin{bmatrix} l & m & 0 \\ -m & l & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where

$$l = \frac{y_{43}}{L},\ \ m = -\frac{x_{43}}{L},\ \ L = \sqrt{x_{43}^2 + y_{43}^2},\ \ x_{ij} = x_i - x_j.$$

The relative displacement on the intersection boundary will be:

$$\delta_{<ab>} = \sum \mathbf{R}_{<ab>}^{(e)}\mathbf{U}_{<ab>}^{(e)} \tag{26}$$

and the matrix form for relative displacement is:

$$\delta_{<ab>} = \mathbf{Z}_M\mathbf{B}_{<ab>}\mathbf{U}_{<ab>}, \tag{27}$$

where

$$\mathbf{B}_{<ab>}^{(e)} = \left\lfloor \mathbf{R}_{<ab>}^{(a)}\mathbf{N}^{(a)}\mathbf{R}_{<ab>}^{(b)}\mathbf{N}^{(b)} \right\rfloor,\ \ \mathbf{U}_{<ab>} = \left\lfloor \mathbf{U}^{(a)}\mathbf{U}^{(b)} \right\rfloor.$$

4. Discretization for HPM

4.1. Lagrange multiplier and penalty function. Physical meaning of the Lagrange multiplier λ is equal to the surface force on the intersection boundary. Generally, in a hybrid-type variational principle, this multiplier is dealt with as an unknown parameter.

Since it has the meaning that Lagrange multiplier λ is the surface force on the boundary $\Gamma_{<ab>}$ in sub-domain $\Omega^{(a)}$ and $\Omega^{(b)}$, the surface force is defined as follows:

$$\lambda_{<ab>} = \mathbf{k} \cdot \delta_{<ab>}. \tag{28}$$

Here $\delta_{<ab>}$ shows relative displacement on the sub-domain boundary $\Gamma_{<ab>}$, and it is shown in three dimensional problem (also plate bending problem) as follows:

$$\left\{ \begin{array}{l} \lambda_{n<ab>} \\ \lambda_{sx<ab>} \\ \lambda_{sy<ab>} \end{array} \right\} = \left[\begin{array}{lll} k_n & 0 & 0 \\ 0 & k_{sx} & 0 \\ 0 & 0 & k_{sy} \end{array} \right] \left\{ \begin{array}{l} \delta_{n<ab>} \\ \delta_{sx<ab>} \\ \delta_{sy<ab>} \end{array} \right\}, \tag{29}$$

where

$$k_n = k_{sx} = k_{sy} = p,$$

where p is a penalty function.

4.2. Discretization for subsidiary condition. The (16) expression we can write by following way:

$$\begin{gathered} \mathbf{H}_{<ab>} = -\delta \int_{\Gamma_{<ab>}} \lambda^t_{<ab>} (\mathbf{u}^{(a)}_{<ab>} - \mathbf{u}^{(b)}_{<ab>}) d\Gamma = \\ = -\delta \int_{\Gamma_{<ab>}} \delta^t_{<ab>} \mathbf{k}_{<ab>} \delta_{<ab>} d\Gamma = -\delta \mathbf{U}^t_{<ab>} \int_{\Gamma_{xy<ab>}} \mathbf{B}^t_{<ab>} \bar{\mathbf{k}}_{<ab>} \mathbf{B}_{<ab>} d\Gamma_{xy} \mathbf{U}_{<ab>}. \end{gathered} \tag{30}$$

Where

$$\mathbf{U}_{<ab>} = \mathbf{M}_{<ab>} \mathbf{U},$$

here M is a matrix which relates the total degree of freedom and the degree of freedom of each sub-domain. It is similar for virtual displacement:

$$\delta \mathbf{U}_{<ab>} = \mathbf{M}_{<ab>} \delta \mathbf{U}.$$

Then we obtain following equation:

$$\mathbf{H}_{<ab>} = -\delta \mathbf{U}^t \mathbf{K}_{<s>} \mathbf{U}, \tag{31}$$

where

$$\mathbf{K}_{<s>} = \mathbf{M}^t_{<s>} \int_{\Gamma_{xy<ab>}} \mathbf{B}^t_{<s>} \bar{\mathbf{k}}_{<s>} \mathbf{B}_{<s>} d\Gamma_{xy} \mathbf{M}_{<s>},$$

where

$$\bar{\mathbf{k}}_{<s>} = \int_{-t/2}^{t/2} \mathbf{Z}^t_M \mathbf{k} \mathbf{Z}_M dZ = \left[\begin{array}{lll} \frac{t^3}{12} k_n & 0 & 0 \\ 0 & \frac{t^3}{12} k_{sx} & 0 \\ 0 & 0 & t k_{sy} \end{array} \right].$$

4.3. Discretization virtual work equation for each sub-domain. For thin plate theory a reduced form of the constitutive relations is obtained by making $\sigma_z = 0$ and subsequently eliminating ε_z. Strains in thin plate are:

$$\begin{gathered}\varepsilon_x = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2},\ \ \varepsilon_y = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2},\\ \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = -2z\frac{\partial^2 w}{\partial x \partial y},\ \ \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0,\ \ \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0.\end{gathered} \tag{32}$$

After application of thin plate strains we obtain $\bar{D}$ matrix for an elastic isotropic material:

$$\bar{\mathbf{D}} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}. \tag{33}$$

Next we are bringing strain vector into the matrix using displacement field.

$$\bar{\varepsilon}^{(e)} = \mathbf{Lu}^{(e)} = \mathbf{ZB}^{(e)}\mathbf{U}^{(e)}, \tag{34}$$

where

$$\mathbf{B}^{(e)} = \mathbf{LN}^{(e)},$$

we will have:

$$\int_{\Omega^{(e)}} [\mathbf{L}\delta\mathbf{u}]^t \sigma d\Omega = \int_{\Omega^{(e)}} \delta\bar{\varepsilon}^{(e)}\bar{\mathbf{D}}^{(e)}\bar{\varepsilon}^{(e)} d\Omega = \left(\delta\mathbf{U}^{(e)}\right)^t \int_{\Omega_{xy}^{(e)}} \mathbf{B}^{(e)t}\mathbf{k}^{(e)}\mathbf{B}^{(e)} d\Omega_{xy}\mathbf{U}^{(e)}. \tag{35}$$

We have:

$$\mathbf{U}^{(e)} = \mathbf{A}^{(e)}\mathbf{U},\ \ \delta\mathbf{U}^{(e)} = \mathbf{A}^{(e)}\delta\mathbf{U}.$$

As mentioned above, it obtains the following:

$$\int_{\Omega^{(e)}} [\mathbf{L}\delta\mathbf{u}]^t \sigma d\Omega = \delta\mathbf{U}\mathbf{K}^{(e)}\mathbf{U} \tag{36}$$

here $\mathbf{A}^{(e)}$ is a matrix which relates the total degree of freedom and the degree of freedom of each sub-domain.

$$\mathbf{K}^{(e)} = \int_{\Omega_{xy}^{(e)}} \mathbf{B}^{(e)t}\mathbf{k}^{(e)}\mathbf{B}^{(e)} d\Omega_{xy}$$

here

$$\mathbf{k}^{(e)} = \int_{-t/2}^{t/2} \mathbf{Z}^t\bar{\mathbf{D}}^{(e)}\mathbf{Z} dZ = \frac{t^3}{12}\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}. \tag{37}$$

The discretization of the body and surface force is as follows:

$$\int_{\Omega^{(e)}} \delta\mathbf{u}^t\mathbf{f} d\Omega + \int_{\Gamma^{(e)}} \delta\mathbf{u}^t\mathbf{T} d\Omega = \delta\mathbf{U}^t\mathbf{P}^{(e)}, \tag{38}$$

where

$$\mathbf{P}^{(e)} = \left(\mathbf{A}^{(e)}\right)^t \int_{\Omega^{(e)}} \left(\mathbf{N}^{(e)}\right)^t \mathbf{Z}_M \mathbf{f}\, d\Omega + \int_{\Gamma^{(e)}} \left(\mathbf{N}^{(e)}\right)^t \mathbf{Z}_M \mathbf{T} d\Gamma.$$

Finally, we obtain following discretized equation:

$$\delta\mathbf{U}^t \left(\sum \mathbf{K}^{(e)} + \sum \mathbf{K}_{<s>}\right) \mathbf{U} - \delta\mathbf{U}^t \left(\sum \mathbf{P}^{(e)}\right) = 0. \tag{39}$$

Since $\delta\mathbf{U}$ is arbitrary, we can write Equation (39) as follows:

$$\mathbf{KU} = \mathbf{P}, \tag{40}$$

where

$$\mathbf{K} = \sum \mathbf{K}^{(e)} + \sum \mathbf{K}_{<s>}, \quad \mathbf{P} = \sum \mathbf{P}^{(e)} .$$

The discretization equation of this model becomes a simulteniuos linear equation shown in equation (40). Left coefficient matrix $\mathbf{K}$ consists of stiffness in the sub-domain and subsidiary condition on the intersection boundary for the adjacent sub-domain. It can express the discontinuous phenomenon of hinge etc., without changing degree of freedom by changing the value of k of equation (28) to zero.

5. Numerical example.

As numerical examples, we present some simple problems.

At first, we give both-end fixed beam with uniform distributed load. The beam has the following material properties and geometrical properties:

Young's Modulus = 1x10^6 kN/m^2 , Poison's ratio = 0, Length=4m, width=1m and thickness=0.1m, Uniform-distributed-load=1kN/m^2.

It has given comparison between exact solution and HPM results. The results for this case have given by the figures 5 and 6. Here for the moment analytical and numerical solutions are equal.

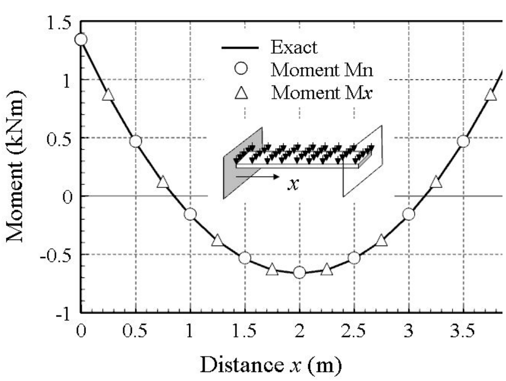

Fig. 5. Moments

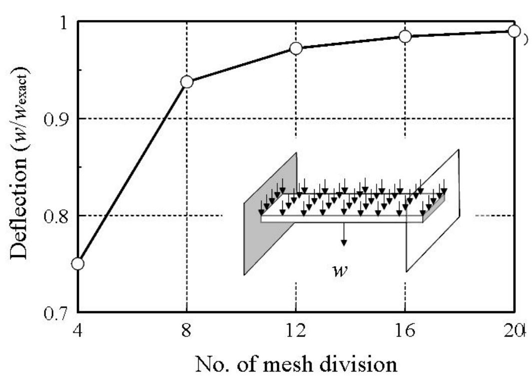

Fig. 6. Ratio of deflections by exact and HPM

The next example is fixed supported circle plate with distributed load. Plate material and geometrical properties are following:

Young's Modulus = $1\text{x}10^6$ kN/m^2 , Poison's ratio = 0, Radius=1m, thickness=0.1m, Distributed Load=1kN/m^2.

In the figure 7, it has also given correspondently contours of moment distribution for xx and xy components and mesh division for circle plate.

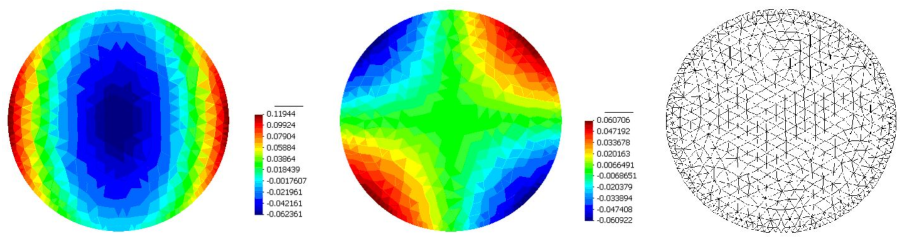

Fig.7 Contours of Moment-xx and Moment-xy distribution and Mesh Division

Also as shown in figure 8 numerical result of the deflection is high accuracy.

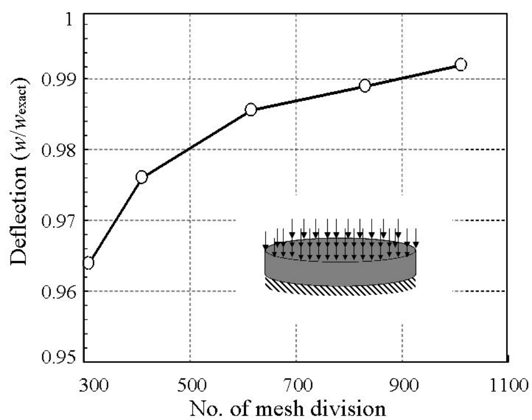

Fig.8 Ratio of deflections by exact and HPM

Next example is simple-supported rectangular plate with concentrated load. The plate properties are following:

Young's Modulus = $1\text{x}10^6$ kN/m^2 , Poison's ratio = 0, Length=4m, width=4m and thickness=0.1m, Concentrated-load=4kN.

In figure 9 and 10, it show the results obtained for this case. Here we obtain high accuracy between exact and HPM results for moment calculations, but not exactly the same solutions.

In figure 11, it has given correspondently contours of moment distribution for xx and xy components and mesh division for simple supported plate with concentrated load.

The last example is again simple-supported plate, but with uniform distributed force, which is equal $1kN/m^2$. Material and geometrical characteristics are the similar with previews example. Results for this case are given in figures 12 and 13.

In figure 14, it has given correspondently contours of moment distribution for xx and xy components and mesh division for this case.

6. Discrete limit analysis

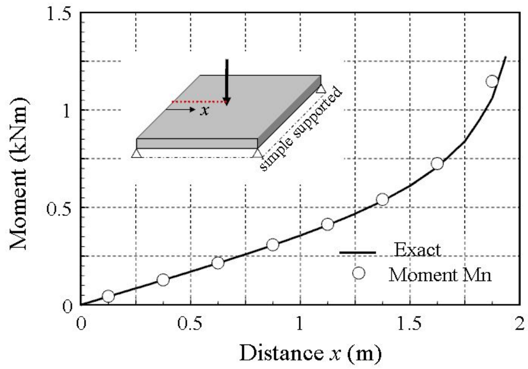

Fig. 9.Moments

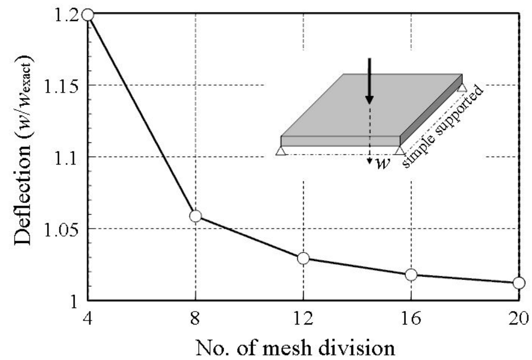

Fig. 10. Ratio of deflections by exact and HPM

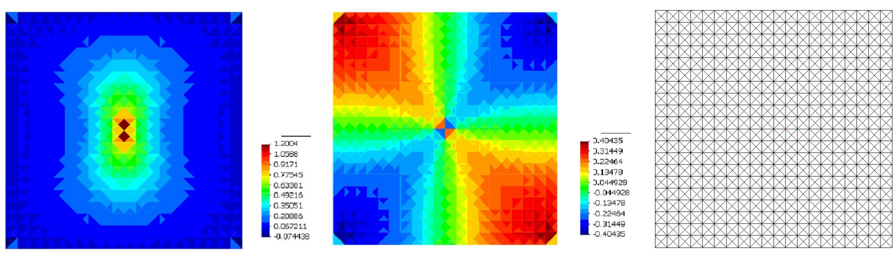

Fig.11 Contours of Moment-xx and Moment-xy distribution and Mesh Division

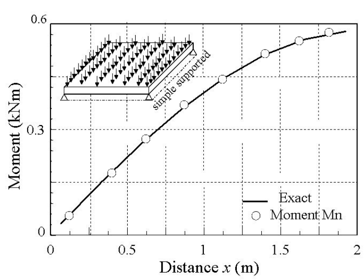

Fig. 12. Moments

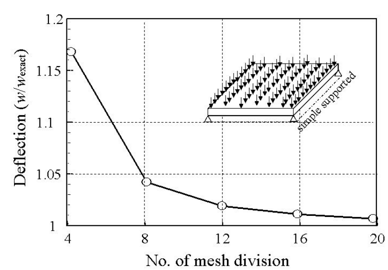

Fig. 13.Ratio of deflections by exact and HPM

6.1. Constitutive Low: In case of plate bending problems yield function has following forms:

$$f(M) = \left(\frac{M_n}{M_{pn}}\right)^2 - 1. \tag{41}$$

If a plastic hinge will occur, the bending moment on intersection boundary is assumed to be zero:

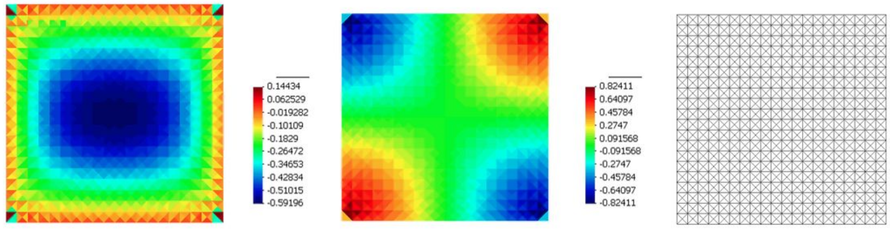

Fig.14 Contours of Moment-xx and Moment-xy distribution and Mesh Division

$$f(M_n) = 0. \tag{42}$$

For this case we can obtain incremental bending moment as follows:

$$\Delta M_n = k^{ep} \Delta \delta, \tag{43}$$

where $\Delta \delta$ is the relative displacement and

$$k^{ep} = \left(k^e - \frac{k^e \frac{\partial f}{\partial \lambda} \frac{\partial Q}{\partial \lambda} k^e}{\frac{\partial f}{\partial \lambda} k^e \frac{\partial Q}{\partial \lambda}} \right). \tag{44}$$

6.2. Load Incremental Method: Load at the (i+1)-th step can be calculated by using the load at the i-th step:

$$P^{(i+1)} = (1 - r_i) P^{(i)}, \tag{45}$$

where r_i is a rate of load increment which we can calculate using this equation:

$$f\left(M_n + r\Delta M_n\right) = 0. \tag{46}$$

After solving following equation:

$$\left(\frac{M_n + r\Delta M_n}{M_{pn}} \right)^2 - 1 = 0$$

we will obtain r :

$$r = \frac{M_{pn} - M_n}{\Delta M_n}. \tag{47}$$

In case of bending moment, residual load at the n-th step will be:

$$P^{(n)} = \prod_{i=0}^{n-1} [(1 - r_i)] P. \tag{48}$$

Cumulative rate of load increment is as follows:

$$r_{TOTAL} = \sum_{k=1}^{n} \left(\prod_{i=0}^{k-1} [(1 - r_i)] \right) r_k. \tag{49}$$

When $r_{TOTAL} = 1$, iteration is finish.

7. Numerical examples for discrete limit analysis: In case of discrete-limit analysis, we have computed simple supported plate with distributed load. The plate has following geometrical and material properties: Young's Modulus = 1x10^6 kN/m^2 , Poison's ratio = 0, Yield Moment: $M_{pn} = 0.1$Nm, Length=2m,width=2m and thickness=0.1m, Distributed Load=1kN/m^2.

As a result of calculation, it has obtained Load-Displacement curve, which has compared with exact solution and as we can see bellow in the graph numerical and exact solutions are the same:

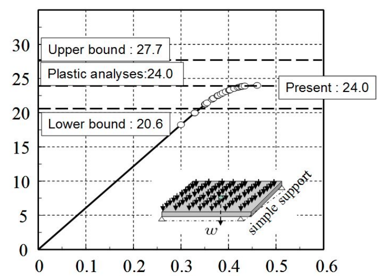

Fig. 15. Load-Displacement Curve

Also it has done comparison between numerical and analytical plastic hinge and we can see bellow for two cases obtained the same results:

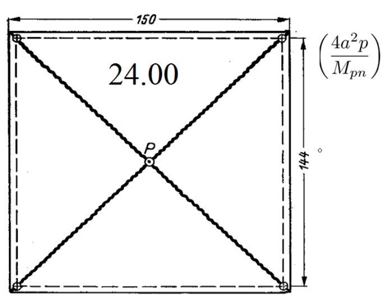

Fig. 16. Theoretical Hinge-line

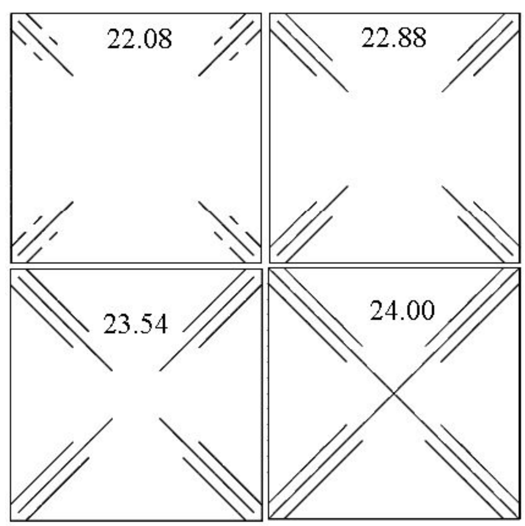

Fig. 17. Numerical Hinge-line

Next example is again simple supported plate with same properties, only now applied concentrated load, which is equal 4kN. For this case, also it has done the same calculations and the same comparisons and in this case also theoretical and numerical results are congruent.

Below given Load-Displacement curve obtained analytically and numerically.

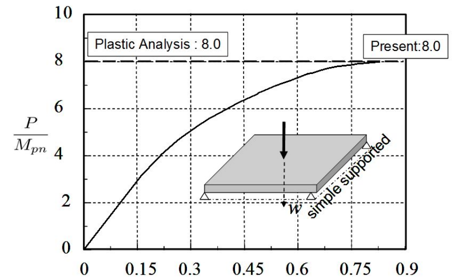

Fig. 18. Load-Displacement Curve Ratio of deflections by exact and HPM

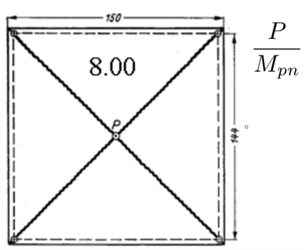

Fig. 19. Theoretical Hinge-line

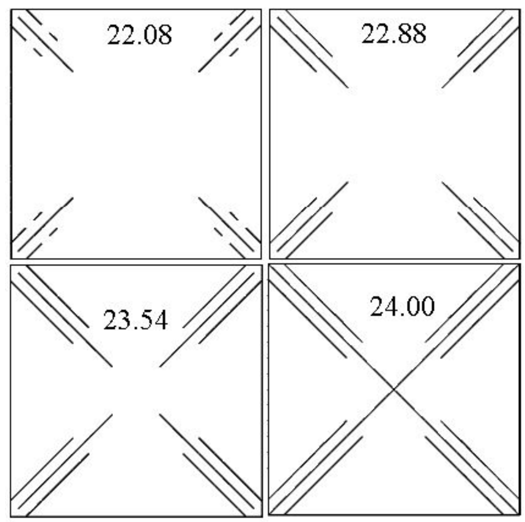

Fig. 20. Numerical Hinge-line

Next it has given hinge lines obtained analytically and numerically.

8. Conclusions. In this paper proposed new approach for solving plate bending problem by using HPM. After comparison of analytical solution and HPM results for deflection and bending moment in case of several examples we can see that we have high accuracy between them. Numerical results for limit load equal to the solutions of plastic analysis. And we can get same collapse pattern about theoretical assumption. As a result, we can conclude that HPM corresponds to all requirements for solving the elastic or elasto-plastic problems such as plate bending.

REFERENCES

[1] *Arnold, D. N.* An interior penalty finite element method with discontinuous elements / D. N. Arnold // SIAM journal on numerical analysis. – 1982. – No. 4. – Vol. 19. – P. 742-760.

[2] *Hughes, T. J. R. K.* On the continuous/discontinuous Galerkin(CDG) formulation of Poisson-Kirkhoff plate theory / Hughes T. J. R ., Garikipati. K. // Computational mechanics-Theory and practice. – 2004.

[3] *Kawai, T.* My challenge in the development of a mixed variational method in solid mechanics / T. Kawai // Applied Mechanics Reviews, Transaction of the ASME. – 2007. – Vol. 60. – P. 51-64.

[4] *Mergheim, J.* A hybrid discontinuous Galerkin/interface method for the computational modeling of failure / J. Mergheim J. et al.// Communications in numerical methods in engineering. – 2004. – Vol. 20. – P. 511-519.

[5] *Ohki, H.* Upper and low bound solution with hybrid-type penalty method / Ohki H., Takeuchi N. // Transactions of JSCES. – 2006. – Paper No 20060020 – P. 1-10.

[6] *Takeuchi, N.* Material nonlinear analysis by using discrete model applied penalty method in hybrid displacement model / N. Takeuchi // Transactions of JSCE. 2001. – Paper № 20010002. – P. 52-62.

[7] *Washidzu, K.* Variational methods in elasticity and plasticity / K. Washidzu. – New York : Pergamon Press, 1968. – P. 238.

Н. Такеучи,А. В. Варданян

УПРУГОПЛАСТИЧЕСКИЙ АНАЛИЗ ИЗГИБА ПЛАСТИН С ИСПОЛЬЗОВАНИЕМ МЕТОДА ПЕНАЛЬТИ ГИБРИДНОГО ТИПА

Хосей университет, Токио, Япония

Институт механики Национальной Академии наук Армении

Аннотация. В работе рассматривается изгиб пластины численным пенальти методом гибридного типа (HPM-Hybrid-type Penalty Method). Данный метод учитывает в линейных и нелинейных смещениях жесткое смещение, жесткое кручение, деформацию и градиент деформации в каждой подобласти, и описывает дополнительные условия непрерывности смещений в рамках вариационных выражений с лагранжевыми множителями. Вводится пенальти функция, описывающая жесткую пружину, соединяющую каждую подобласть. Шарнирные линии вычисляются с использованием нелинейности в функциях пенальти как пружинных систем. Приводятся примеры решения задач изгиба пластин численным пенальти методом гибридного типа

Ключевые слова: изгиб пластины, метод пенальти, виртуальная работа гибридного типа, разрывный метод Галеркина

Такеучи Норио

доктор наук, профессор, Хосей университет, Токио, Япония

Варданян Анна Ваниковна

аспирант, Институт механики НАН Армении, Ереван

Takeuchi Norio

doctor of sciences, professor, Department of Art and Technology, Hosei University, Tokyo, Japan

`e-mail:` takeuchi@hosei.ac.jp

Vardanyan Anna Vanikovich

post-graduate student, Institute of Mechanics, Armenian National Academy of Sciences

`e-mail:` vardanyan_a@yahoo.com