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МАТРИЧНЫЙ МЕТОД ПОЛУЧЕНИЯ ПОЛИНОМИАЛЬНЫХ РЕШЕНИЙ ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ В ЧАСТНЫХ ПРОИЗВОДНЫХ С ПОСТОЯННЫМИ КОЭФФИЦИЕНТАМИ

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Аннотация. Разрабатывается матричный метод для конструктивного определения полиномиальных решений линейных дифференциальных уравнений в частных производных (ДУЧП) с постоянными коэффициентами. Отметим, что наш метод применим также, если полиномы, которые индуцируют ДУЧП, имеют постоянные слагаемые (аналогично уравнению Гельмгольца), и, следовательно, такие ДУЧП не могут иметь чисто полиномиальных решений. В этом случае разрабатываемый матричный метод обеспечивает полиномиальными решениями, умноженными на экспоненты. Более того, метод позволяет найти полиномиальное (умноженные на экспоненту) решение ДУЧП с полиномиальной (умноженной на экспоненту) правой частью. Также метод сокращает затраты на построение полиномиального (умноженного на экспоненциальное) решение ДУЧП для определения нулевого пространства дифференциального оператора алгебраической блочно-матричной линейной системы (с числовыми записями). Кроме того, используя матричный подход, можно исследовать некоторые алгебраический свойства, такие как размерность и базис пространства полиномиальных решений (в общем случае, умноженных на экспоненты). В частности, для пространства полиномиальных решений мы можем решить задачу с точностью до некоторой сколь угодно большой степени. В частности, мы обобщаем задачу о бесконечной степени полиномиальных решений многочлена ДУЧП на экспоненциальный случай. Более того, ДУЧП может содержать ненулевую правую часть многочлена (умноженную на экспоненту). Рассматриваются некоторые примеры полиномиальных решений (в общем случае умноженных на экспоненты) уравнений Лапласа, Гельмгольца и Пуассона.

Ключевые слова: системы линейных дифференциальных уравнений в частных производных, дифференциальные уравнения с постоянными коэффициентами, неоднородные дифференциальные уравнения, полиномиальное решение, экспоненциальное решение, нуль-пространство матрицы

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Research Article

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MATRIX METHOD OF POLYNOMIAL SOLUTIONS TO CONSTANT COEFFICIENT PDE'S

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Abstract. In the paper, we introduce a matrix method to constructively determine polynomial (in general, multiplied by exponentials) solutions to the constant coefficient linear Partial Difference Equations (PDE's). Note our method also is applicable if the polynomials that induce PDE's have constant terms (similarly Helmholtz' equation) and consequently such PDE's cannot have pure polynomial solutions. In this case our matrix method supplies polynomial solutions multiplied by exponentials. Moreover, the method allows to find a polynomial (multiplied by an exponential) solution to PDE with polynomial (multiplied by the exponential) right-hand side. Our matrix method reduces the funding of polynomial (multiplied by an exponential) solution to PDE's to determine the differential operator null-space of algebraic block matrix linear system (with numerical entries). Furthermore, using our matrix approach, we can investigate some linear algebra properties such as dimension and basis of the space of polynomial solution (generally, multiplied by an exponential). In particular, for a polynomial solution space we can decide the problem up to some arbitrarily large degree, In particular, we generalize problem about infinite degree of polynomial solutions of polynomial PDE to exponential case. Moreover, PDE can contain a nonzero polynomial (multiplied by an exponential) right-hand side. Some examples of polynomial solutions (multiplied by exponentials, in general) to the Laplace, Helmholtz, and Poisson equations are considered.

Keywords: polynomial solution to linear constant coefficient PDE's, exponential solution to non homogeneous PDE, null-space of matrix.

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Introduction. The polynomial solutions to linear constant coefficient PDE('s) is the well-known problem of algebra; see, for example, [1–6]. The most part of the methods to find polynomial solutions to PDE's is based on complicated general algebra approaches (like a primary decomposition).

Certainly, substituting a polynomial with *unknown* (constant) coefficients to PDE('s) and equaling the result to zero, we obtain a linear algebraic system that the solution of the system defines constant coefficients of the polynomial(s). However, this solution of the problem is very one-sided: in general, we do not know existence, dimension, basis of the polynomial solution.

To find polynomial solutions to linear constant coefficient PDE's we offer a matrix method. The matrix method was stimulated by a generalization of the Strang-Fix conditions, see [7, 8]. Our method enables to determine some (linear algebra) characteristics of a solution space such as dimension, basis, affine-invariance, maximal total degree of polynomials, etc.

Note our matrix method is applicable if the polynomials that induce PDE's have constant terms. In the case of constant terms the polynomials that are solutions to PDE's must be multiplied by exponential(s).

Moreover, our matrix method allows to solve PDE with polynomial (multiplied by an exponential, in general) right-hand side.

The polynomial solutions (multiplied by exponentials) to the well-known differential equations (like Laplace's equation), when we take a root of the symbol of differential operator that the root is not the origin, see also [9], can be obtained. So, the shift of a differential equation is equivalent to the multiplication, by the corresponding exponent, of a polynomial solution of the differential equation.

It is well know, see [1], that the degree of the polynomial that is a solution to constant coefficient linear PDE is arbitrary large. And our method allows to generalize this result to polynomials multiplied by exponentials.

Note that the matrix method is valid for polynomials that induce PDE's with coefficients from any algebraically closed field. Moreover, the method can be directly algorithmized.

The paper is organized as follows. Section 1 contains used in the paper notations and definitions. In particular, in Subsection 1.2, the lexicographically ordered sets of monomials and derivatives are introduced. Section 2 is devoted to the matrix of the linear system; in Subsections 2.1, a method to construct the matrix is presented, and, in Subsection 2.2, some properties of the matrix are discussed. In Section 3, the matrix method to solve (in particular, induced by inhomogeneous polynomials) PDE's is discussed. Moreover, in Subsection 3.3, the matrix method to solve PDE with a polynomial right-hand side is considered. Section 4 is devoted to polynomial solutions to some PDE's.

1. Notations and definitions

1.1. Basic notations. Let $d \in \mathbb{N}$, $d \geq 2$, be the number of independent variables.

Let I be the *identity operator*. Let $\delta_{ij} := \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j \end{cases}$ be the *Kronecker delta* and δ be the *Dirac delta-distribution*.

A *multi-index* α is a d-tuple $(\alpha_1, \dots, \alpha_d)$ with its components being nonnegative integers, i. e., $\alpha \in \mathbb{Z}^d_{\geq 0}$. The *length* of a multi-index $\alpha := (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d_{\geq 0}$ is defined as $\alpha_1 + \dots + \alpha_d$ and denoted by $|\alpha|$. For multi-indices $\alpha := (\alpha_1, \dots, \alpha_d)$, $\beta := (\beta_1, \dots, \beta_d)$, we write $\beta \leq \alpha$ if $\beta_j \leq \alpha_j$ for all $j = 1, \dots, d$. The *factorial* of $\alpha := (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d_{\geq 0}$ is $\alpha! := \alpha_1! \cdots \alpha_d!$. The *binomial coefficient* for multi-indices α, β is

$$\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d} = \frac{\alpha!}{\beta!(\alpha - \beta)!}.$$

By definition, put

$$\binom{\alpha}{\beta} = 0 \qquad \text{if } \beta \not\leq \alpha. \tag{1}$$

By x^α, where $x := (x_1, \dots, x_d)$, $\alpha := (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d_{\geq 0}$, denote a monomial $x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. Note that the *total degree* of x^α is $|\alpha|$. By Π_l, $l \in \mathbb{Z}_{\geq 0}$, denote the space of (homogeneous) polynomials that the total degree of the polynomials is equal to l: $\Pi_l := \operatorname{span}\left\{x^\alpha : \alpha \in \mathbb{Z}^d_{\geq 0}, |\alpha| = l\right\}$; by $\Pi_{\leq l}$ denote the space of polynomials that the total degree of the polynomials is less than or equal to l: $\Pi_{\leq l} := \operatorname{span}\left\{x^\alpha : \alpha \in \mathbb{Z}^d_{\geq 0}, |\alpha| \leq l\right\}$.

Remark 1.1. Since the linear algebra definitions and assertions are valid for *any* field; we can consider polynomials with coefficients from an arbitrary field. On the other hand, we prefer to use algebraically closed fields (a field $\mathbb{C}$, for example) or we must use algebraic extensions of the fields.

So, above and in the sequel, 'span' means the linear span over $\mathbb{C}$; and by Π we denote *all* polynomials of d variables with constant coefficients from $\mathbb{C}$.

The dot product of two vectors (d-tuples) $x := (x_1, \dots, x_d)$, $y := (y_1, \dots, y_d)$ is $x \cdot y := x_1 y_1 + \dots + x_d y_d$. If all the polynomials from the space Π_l multiplied by an exponential $e^{ix_0 \cdot x}$, where $x_0 \in \mathbb{C}^d$ is a given point; then we shall write $e^{ix_0 \cdot x}\Pi_l$ (for $\Pi_{\leq l}$, $e^{ix_0 \cdot x}\Pi_{\leq l}$).

Let D^α imply a differential operator $D_1^{\alpha_1} \cdots D_d^{\alpha_d}$, where D_n, $n = 1, \dots, d$, is the partial derivative with respect to the nth coordinate. Note that $D^{(0,\dots,0)}$ is the identity operator. Abusing notations, for a function $f = f(x)$ and *constant* point x_0 we shall write everywhere $D^\alpha f(x_0)$, meaning, in fact, $D^\alpha f(x)|_{x=x_0}$.

The multi-dimensional version of the *Leibniz rule* is

$$(fg)^{(\alpha)} = \sum_{\substack{\beta \in \mathbb{Z}^d_{\geq 0} \\ \beta \leq \alpha}} \binom{\alpha}{\beta} f^{(\beta)} g^{(\alpha - \beta)}, \quad \alpha \in \mathbb{Z}^d_{\geq 0}, \tag{2}$$

where the functions $f(x)$, $g(x)$, $x := (x_1, \dots, x_d)$, are sufficiently differentiable.

The Fourier transform $\mathfrak{F}$ of a function $f \in L^1\left(\mathbb{R}^d\right)$ is defined by

$$f(x) \mapsto \hat{f}(\xi) = (\mathfrak{F}f)(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i\xi \cdot x}\, dx, \qquad \xi \in \mathbb{R}^d.$$

By S' denote the *space of tempered distributions*; and note that the Fourier transform can be extended to (compactly supported) distributions from $S'\left(\mathbb{R}^d\right)$. Moreover, the domain of the Fourier transform can be extended (it is possible, in particular, for compactly supported functions) to the whole complex space $\mathbb{C}^d$. And the Schwartz space S (of test functions), i. e., the space of functions that all the derivatives of the functions are rapidly decreasing, also can be extended to $\mathbb{C}^d$.

So, for $x_0 \in \mathbb{C}^d, \alpha \in \mathbb{Z}^d_{\geq 0}$, we have the following formula

$$\left(\mathfrak{F} e^{ix_0\cdot x} x^\alpha\right)(\xi) = i^{|\alpha|} D^\alpha \delta(\xi - x_0), \qquad \xi \in \mathbb{C}^d.$$

Definition 1.1. Let $f, g \in L^2\left(\mathbb{C}^d\right)$ be complex functions. Then an inner product in the space $L^2\left(\mathbb{C}^d\right)$ is

$$\langle f|g\rangle := \int_{\mathbb{C}^d} f(x)\overline{g(x)}\,dx. \tag{3}$$

Here and in the sequel, the overline $\overline{\cdot}$ denotes the complex conjugation.

In the following definition, we consider complex distributions and complex test functions, see for example [10, 11].

Definition 1.2. Let $\phi \in S(\mathbb{C}^d)$ be a *complex* test function. Let $f = f(x)$, $x \in \mathbb{C}^d$, be a locally integrable on $\mathbb{C}^d$ *complex* function. Then the function f induces a distribution T_f (continuous linear functional) on $S(\mathbb{C}^d)$ as follows

$$T_f(\phi) := \int_{\mathbb{C}^d} \overline{f(x)}\phi(x)\,dx = \overline{\langle f|\phi\rangle}, \tag{4}$$

where $\langle\cdot|\cdot\rangle$, in the right-hand side of (4), is the inner product defined by (3).

Any functional defined by (4) is a *linear functional*; in particular, the functional is *homogeneous*:

$$T_f(a\phi) = aT_f(\phi),$$

where a is a complex valued function.

In the paper, we usually denote matrices by upper-case bold symbols and (sometimes) enclose the symbols of matrices in the square brackets. On the other hand, abusing notation slightly, we shall denote a vector of some linear space by a plain lower-case symbol and interpret the vector as a column vector.

By $\mathbf{I}_n$, $n \in \mathbb{N}$, denote the $n \times n$ *identity matrix*.

Now recall some block matrix notions. A *block matrix* is a matrix broken into sections called *blocks* or *submatrices*. A *block diagonal matrix* is a block matrix such that the main diagonal submatrices can be non-zero and all the off-diagonal submatrices are zero matrices. The (block) diagonals can be specified by an index k measured relative to the main diagonal, thus the main diagonal has $k = 0$ and the k-diagonal consists of the entries on the kth diagonal above the main diagonal. Note that the diagonal submatrices (excepting the main block diagonal) can be non-square.

1.2. Ordered sets By $<_{\text{lex}}$ we denote some *lexicographical order* and by $\mathcal{A}_k$, $k \in \mathbb{Z}_{\geq 0}$, denote the *lexicographically ordered set of all multi-indices of length k*

$$\mathcal{A}_k := \left({}^1\alpha, {}^2\alpha, \dots, {}^{d(k)}\alpha\right), \qquad \begin{array}{l} {}^q\alpha \in \mathbb{Z}^d_{\geq 0},\ |{}^q\alpha| = k,\ q = 1, \dots, d(k), \\ {}^q\alpha <_{\text{lex}} {}^{q'}\alpha \iff q < q', \end{array}$$

where

$$d(k) := \binom{d+k-1}{k} = \frac{(d+k-1)!}{k!(d-1)!}$$

is the number of k-combinations with repetition from the d elements.

By $\widetilde{\mathcal{A}}_k$ we denote a *concatenated set of multi-indices*

$$\widetilde{\mathcal{A}}_k := (\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_k),$$

where the comma must be considered as a concatenation operator to join of two sets. Actually the order of $\widetilde{\mathcal{A}}_k$ is the *graded lexicographical order*. By $\widetilde{d}(k)$ denote the length of a concatenated set like $\widetilde{\mathcal{A}}_k$

$$\widetilde{d}(k) := d(0) + d(1) + \dots + d(k) = \frac{(d+k)!}{k!d!}.$$

By $\mathcal{P}_k$, $k \in \mathbb{Z}_{\geq 0}$, denote the *lexicographically ordered set of all monomials of total degree k*

$$\mathcal{P}_k(x) := \left(x^{{}^1\alpha}, \dots, x^{{}^{d(k)}\alpha}\right), \qquad x = (x_1, \dots, x_d),\ \left({}^1\alpha, \dots, {}^{d(k)}\alpha\right) = \mathcal{A}_k.$$

Remark 1.2. In examples (see Subsection 2.1 and Section 4), we shall use the obvious order of variables: $x > y > z$ ($\zeta > \eta > \theta$, for the Fourier space variables).

By $\mathcal{D}_k$ denote the *ordered set of differential operators*

$$\mathcal{D}_k := \left((-i)^k D^{{}^1\alpha}, \dots, (-i)^k D^{{}^{d(k)}\alpha}\right), \qquad \left({}^1\alpha, \dots, {}^{d(k)}\alpha\right) = \mathcal{A}_k, \tag{5}$$

and, for $\beta \in \mathbb{Z}^d_{\geq 0}$, by $\mathcal{D}^\beta_k$ denote the following set of operators

$$\mathcal{D}^\beta_k := \left((-i)^{k-|\beta|}\binom{{}^1\alpha}{\beta} D^{{}^1\alpha-\beta}, \dots, (-i)^{k-|\beta|}\binom{{}^{d(k)}\alpha}{\beta} D^{{}^{d(k)}\alpha-\beta}\right), \qquad \left({}^1\alpha, \dots, {}^{d(k)}\alpha\right) = \mathcal{A}_k. \tag{6}$$

Remark 1.3. If sets (5), (6) are applied to some function f, then these applications are distributive over comma: $\mathcal{D}_k f := \left((-i)^k D^{{}^1\alpha} f, \dots, (-i)^k D^{{}^{d(k)}\alpha} f\right)$.

Note that if, for some $q \in \{1, \dots, d(k)\}$, $\beta \not\leq {}^q\alpha$; then the qth entry of (6) is zero. Moreover, if $|\beta| > k$; then set (6) is zero set.

Note also that the zero entries of row-vector (6) can be interpreted as zeroizing.

By $\widetilde{\mathcal{P}}_k$ denote the following *concatenated set of monomials*

$$\widetilde{\mathcal{P}}_k := (\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_k). \tag{7}$$

The *concatenated set of derivatives* is defined similarly to (7)

$$\widetilde{\mathcal{D}}_k := (\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_k). \tag{8}$$

2. The matrix of the linear system

2.1. Formation of the matrix. For some $k, l \in \mathbb{Z}_{\geq 0}$, $k \leq l$, define a $\widetilde{d}(l) \times d(k)$ matrix $\mathbf{D}_k$ as follows

$$\mathbf{D}_k := \begin{bmatrix} \mathbf{D}_k^0 \\ \mathbf{D}_k^1 \\ \vdots \\ \mathbf{D}_k^k \\ \widetilde{d}(l)-\widetilde{d}(k) \left\{ \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \right. \end{bmatrix}, \tag{9}$$

where $\mathbf{D}_k^r$ are $d(r) \times d(k)$, $r = 0, 1, \ldots k$, submatrices defined as

$$\mathbf{D}_k^r := \begin{bmatrix} \left[\mathcal{D}_k^{{}^1\beta}\right] \\ \left[\mathcal{D}_k^{{}^2\beta}\right] \\ \vdots \\ \left[\mathcal{D}_k^{{}^{d(r)}\beta}\right] \end{bmatrix}, \qquad \left({}^1\beta, \ldots, {}^{d(r)}\beta\right) = \mathcal{A}_r, \tag{10}$$

and the row vectors $\left[\mathcal{D}_k^{{}^q\beta}\right]$, $q = 1, \ldots, d(r)$, are given by (6).

Remark 2.1. Note that if $r > k$; then the submatrix $\mathbf{D}_k^r$, defined by (10), is a zero matrix.

Finally, for $l \in \mathbb{Z}_{\geq 0}$, define a $\widetilde{d}(l) \times \widetilde{d}(l)$ matrix $\widetilde{\mathbf{D}}_l$ as

$$\widetilde{\mathbf{D}}_l := \begin{bmatrix} \mathbf{D}_0 & \mathbf{D}_1 & \ldots & \mathbf{D}_{l-1} & \mathbf{D}_l \end{bmatrix} = \begin{bmatrix} \mathbf{D}_0^0 & \mathbf{D}_1^0 & \ldots & \mathbf{D}_{l-1}^0 & \mathbf{D}_l^0 \\ 0 & \mathbf{D}_1^1 & \ldots & \mathbf{D}_{l-1}^1 & \mathbf{D}_l^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \mathbf{D}_{l-1}^{l-1} & \mathbf{D}_l^{l-1} \\ 0 & 0 & \ldots & 0 & \mathbf{D}_l^l \end{bmatrix}. \tag{11}$$

Remark 2.2. Note that, in formulas (9), (11), the symbol '0' must be considered as a zero submatrix of the corresponding size. (See Remarks 1.3.)

The component-wise form of the matrix $\widetilde{\mathbf{D}}_l$, $l \in \mathbb{Z}_{\geq 0}$, is

$$\left[\widetilde{\mathbf{D}}_l\right]_{qr,\ 1\leq q,r\leq \widetilde{d}(l)} = \begin{cases} (-i)^{|{}^r\alpha - {}^q\beta|} \binom{{}^r\alpha}{{}^q\beta} D^{{}^r\alpha - {}^q\beta}, & {}^q\beta \leq {}^r\alpha\,, \\ 0, & \text{otherwise}, \end{cases} \tag{12}$$

where $\left({}^1\alpha, \ldots, {}^{\widetilde{d}(l)}\alpha,\right) = \left({}^1\beta, \ldots, {}^{\widetilde{d}(l)}\beta,\right) = \widetilde{\mathcal{A}}_l$.

Example The bivariate $\widetilde{d}(3) \times \widetilde{d}(3) = 10 \times 10$ matrix $\widetilde{\mathbf{D}}_3$ is of the form:

$$\widetilde{\mathbf{D}}_3 := \left[\begin{array}{c|cc|ccc|cccc}
I & -i\partial_x & -i\partial_y & -\partial_{xx} & -\partial_{xy} & -\partial_{yy} & i\partial_{xxx} & i\partial_{xxy} & i\partial_{xyy} & i\partial_{yyy} \\ \hline
0 & I & 0 & -2i\partial_x & -i\partial_y & 0 & -3\partial_{xx} & -2\partial_{xy} & -\partial_{yy} & 0 \\
0 & 0 & I & 0 & -i\partial_x & -2i\partial_y & 0 & -\partial_{xx} & -2\partial_{xy} & -3\partial_{yy} \\ \hline
0 & 0 & 0 & I & 0 & 0 & -3i\partial_x & -i\partial_y & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & -2i\partial_x & -2i\partial_y & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & -i\partial_x & -3i\partial_y \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I
\end{array}\right].$$

2.2. Some properties of the matrix $\widetilde{\mathbf{D}}_l$

Theorem 2.1. *Let* $l \in \mathbb{Z}_{\geq 0}$. *Let the set* $\widetilde{\mathcal{D}}_l$ *be given by* (8), *the matrix* $\widetilde{\mathbf{D}}_l$ *be given by* (11), *and functions* f, g *be sufficiently differentiable. Then we have*

$$\left[\widetilde{\mathcal{D}}_l(fg)\right] = \left[\widetilde{\mathcal{D}}_l f\right] \widetilde{\mathbf{D}}_l g = \left[\widetilde{\mathcal{D}}_l g\right] \widetilde{\mathbf{D}}_l f. \tag{13}$$

Here we omit the proof of formula (13) and note only that the formula is a direct consequence of form (12) and the Leibniz rule, see (2).

For a single function we investigate the ranks of submatrices in the upper right corner of the matrix $\widetilde{\mathbf{D}}_l f(x_0)$ (the matrices $\mathbf{D}_m^{m'} f(x_0)$, $m' = 0, \ldots, m$, $m = 0, \ldots, l$, that are given by (10)); where the function $f : \mathbb{C}^d \to \mathbb{C}^d$ is sufficiently differentiable and $x_0 \in \mathbb{C}^d$ is a given point.

Proposition 2.2. *Let* $l \in \mathbb{Z}_{\geq 0}$. *The submatrix* $\mathbf{D}_m^{m'}$, $m' = 0, \ldots, m$, $m = 0, \ldots, l$, *contains the derivatives of order* $m - m'$ *only.*

Corollary 2.3. *All the submatrices on the mth,* $m = 0, \ldots, l$, *block diagonal of the matrix* $\widetilde{\mathbf{D}}_l$, *i. e., the submatrices* $\mathbf{D}_m^0, \mathbf{D}_{m+1}^1, \ldots, \mathbf{D}_l^{l-m}$, *contain the derivatives of order* m.

It easy to see that the qth, $1 \leq q \leq d(m)$, row of the matrix $\mathbf{D}_m^m$, $m = 0, \ldots, l$, contains only one non-zero element I, which is situated on the qth position. Consequently, $\mathbf{D}_m^m = \mathbf{I}_{d(m)}$. So, since the matrix $\widetilde{\mathbf{D}}_l f(x_0)$ is an upper triangular matrix, we can state an obvious theorem.

Theorem 2.4. *The matrix* $\widetilde{\mathbf{D}}_l f(x_0)$, $l \in \mathbb{Z}_{\geq 0}$, *is singular iff* $f(x_0) = 0$.

Now state a theorem about ranks of all other blocks of the matrix $\widetilde{\mathbf{D}}_l f(x_0)$.

Theorem 2.5. *The* $d(m') \times d(m)$ *submatrix* $\mathbf{D}_m^{m'} f(x_0)$, $m' = 0, \ldots, m-1$, $m = 1, \ldots, l$, $l \in \mathbb{N}$, *has full rank, i. e., the rank of* $\mathbf{D}_m^{m'} f(x_0)$ *is equal to* $d(m')$, *if and only if there exists at least one non-zero derivative* $D^\gamma f(x_0)$, $|\gamma| = m - m'$.

The proof of Theorem 2.5 is given in A.

Hence we see that each of the submatrices $\mathbf{D}_m^{m'} f(x_0)$, $m' = 0, \ldots, m$, $m = 0, \ldots, l$, is either a full rank matrix or zero matrix.

The following theorem allows to determine the dimension of $\ker \widetilde{\mathbf{D}}_l f(x_0)$.

Theorem 2.6. *Let $l \in \mathbb{Z}_{\geq 0}$, let $f : \mathbb{C}^d \to \mathbb{C}^d$ be a function, and let x_0 be a point of $\mathbb{C}^d$. Let the matrix $\widetilde{\mathbf{D}}_l$ be given by (11). Let $D^\alpha f(x_0)$, $\alpha \in \mathbb{Z}^d_{\geq 0}$, be a non-zero derivative of the least order. Then*

$$\dim \ker \widetilde{\mathbf{D}}_l f(x_0) = \begin{cases} \widetilde{d}(l) - \widetilde{d}(l - |\alpha|) & \text{if } l \geq |\alpha| > 0; \\ \widetilde{d}(l) & \text{if } l < |\alpha|. \end{cases} \tag{14}$$

The theorem is a direct consequence of the following lemma.

Lemma 2.7. *Under the conditions of Theorem 2.6, we have*

$$\operatorname{rank} \widetilde{\mathbf{D}}_l f(x_0) = \begin{cases} \widetilde{d}(l - |\alpha|) & \text{if } l \geq |\alpha| > 0; \\ 0 & \text{if } l < |\alpha|. \end{cases}$$

Sketch of the proof of Lemma 2.7. For the case $l \geq |\alpha| > 0$, by Theorem 2.5 and Corollary 2.3, each block on the $|\alpha|$th block diagonal of the matrix $\widetilde{\mathbf{D}}_l f(x_0)$ is a full rank matrix. However, since the blocks of $\widetilde{\mathbf{D}}_l f(x_0)$ are not, generally, square matrices; the problem to determine $\operatorname{rank} \widetilde{\mathbf{D}}_l f(x_0)$ is not trivial.

Using an analog of the Gaussian elimination algorithm (applied to columns instead of rows) and moving (actually permutating) zero columns to the left, the matrix $\widetilde{\mathbf{D}}_l f(x_0)$ can always be transformed into a *strictly* upper triangular matrix, where the lowest non-zero diagonal goes to the lower right corner of the last submatrix $\mathbf{D}_l^{l-|\alpha|}$. So $\operatorname{rank} \widetilde{\mathbf{D}}_l f(x_0) = \widetilde{d}(l - |\alpha|)$.

Since, in the case $l < |\alpha|$, the matrix $\widetilde{\mathbf{D}}_l f(x_0)$ is a zero matrix; the case $l < |\alpha|$ is trivial. □

Introduce some notation.

Definition 2.1. Let $l \in \mathbb{Z}_{\geq 0}$, let a function $f : \mathbb{C}^d \to \mathbb{C}^d$ be sufficiently differentiable, and let x_0 be a point of $\mathbb{C}^d$. Let the matrix $\widetilde{\mathbf{D}}_l$ be given by (11). By V_l denote the *(right) null-space of the matrix $\widetilde{\mathbf{D}}_l f(x_0)$:*

$$V_l := \ker \widetilde{\mathbf{D}}_l f(x_0). \tag{15}$$

The space $\mathbb{C}^{\widetilde{d}(l)}$, $l \in \mathbb{Z}_{\geq 0}$, can be considered as a space with a Cartesian coordinate system, where the dot product of the Cartesian coordinates is $x \cdot y := \sum_q x_q \overline{y_q}$. Namely, we have

$$\mathbb{C}^{\widetilde{d}(l)} := \operatorname{span} \left\{ e_q : 1 \leq q \leq \widetilde{d}(l) \right\},$$

where the span is over $\mathbb{C}$ and e_q is the qth basis vector: $e_q := \left(\delta_{q1}, \ldots, \delta_{q,\widetilde{d}(l)} \right)$; and we can decompose $\mathbb{C}^{\widetilde{d}(l)}$ as follows

$$\mathbb{C}^{\widetilde{d}(l)} = {}^0\mathbb{C}^{\widetilde{d}(l)} \oplus {}^1\mathbb{C}^{\widetilde{d}(l)} \oplus \cdots \oplus {}^l\mathbb{C}^{\widetilde{d}(l)}, \tag{16}$$

where

$$
\begin{aligned}
&{}^0\mathbb{C}^{\widetilde{d}(l)} := \operatorname{span}\{e_1\},\\
&{}^m\mathbb{C}^{\widetilde{d}(l)} := \operatorname{span}\left\{e_q : \widetilde{d}(m-1)+1 \le q \le \widetilde{d}(m)\right\}, \qquad m = 1,\dots,l,
\end{aligned}
$$

and the direct sums in (16) are orthogonal. Decomposition (16) corresponds to the block structure of the matrix $\widetilde{\mathbf{D}}_l$, $l \in \mathbb{Z}_{\ge 0}$, (as well as structures of $\widetilde{\mathcal{P}}_l$, $\widetilde{\mathcal{D}}_l$).

Definition 2.2. By $\mathfrak{P}_m$ denote the *orthogonal projection* of $\mathbb{C}^{\widetilde{d}(l)}$ onto ${}^m\mathbb{C}^{\widetilde{d}(l)}$, $m = 0,\dots,l$; and define the subspaces of the null-space V_l, see Definition 2.1, as

$$
{}^mV_l := \mathfrak{P}_m V_l, \qquad m = 0,1,\dots,l. \tag{17}
$$

Note that generally V_l is not a sum, like (16), of ${}^mV_l$. In Subsection 3.1, we shall return to this problem in respect to the affine invariance of polynomial spaces.

Now we can formulate the following theorem.

Theorem 2.8. *Let $l \in \mathbb{N}$. Let the matrix $\widetilde{\mathbf{D}}_l f(x_0)$ be singular and the space V_l is defined by* (15). *Then the subspace ${}^lV_l := \mathfrak{P}_l V_l$ is non-zero.*

Proof of Theorem 2.8. If the matrix $\widetilde{\mathbf{D}}_l f(x_0)$ is a zero matrix, there is nothing to prove.

Now let $\widetilde{\mathbf{D}}_l f(x_0)$ be non-zero. Suppose ${}^lV_l$ is a zero space; then

$$
\dim\ker \widetilde{\mathbf{D}}_l f(x_0) = \dim\ker \widetilde{\mathbf{D}}_{l-1} f(x_0). \tag{18}
$$

Since the matrix $\widetilde{\mathbf{D}}_l f(x_0)$ is singular and non-zero; there exists a number $r \in \mathbb{N}$, $1 \le r \le l$, that the block r-diagonal is the lowest non-zero block diagonal. By Theorem 2.6, we have

$$
\begin{aligned}
&\dim\ker \widetilde{\mathbf{D}}_{l-1} f(x_0) = \begin{cases} \widetilde{d}(l-1) - \widetilde{d}(l-1-r) & \text{if } r \le l-1;\\ \widetilde{d}(l-1) & \text{if } r > l-1 \end{cases},\\
&\dim\ker \widetilde{\mathbf{D}}_l f(x_0) = \widetilde{d}(l) - \widetilde{d}(l-r).
\end{aligned}
$$

Nevertheless, for all $r, l \in \mathbb{N}$, $r \le l$, we get

$$
\left(\begin{cases} \widetilde{d}(l-1) - \widetilde{d}(l-1-r) & \text{if } r \le l-1;\\ \widetilde{d}(l-1) & \text{if } r > l-1 \end{cases}\right) < \widetilde{d}(l) - \widetilde{d}(l-r).
$$

This contradicts equality (18). The theorem is proved. □

If we have several functions $f_1, f_2, \dots, f_n$, we must consider a block matrix

$$
\widetilde{\mathbf{D}}_l f(x_0) := \begin{bmatrix} \widetilde{\mathbf{D}}_l f_1(x_0) \\ \vdots \\ \widetilde{\mathbf{D}}_l f_n(x_0) \end{bmatrix}, \tag{19}
$$

where by f we denote the following column vector $\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$.

Definition 2.3. Let $l \in \mathbb{Z}_{\geq 0}$. Similarly to Definition 2.1, by V_l denote the null-space of block matrix (19).

3. Solution method

3.1. One homogeneous PDE

Theorem 3.1. *Let $P \in \Pi$, let $l \in \mathbb{Z}_{\geq 0}$, let x_0 be a point of $\mathbb{C}^d$. Let the matrix $\widetilde{\mathbf{D}}_l$ be given by* (11)*, let the space V_l be given by* (15)*, and let the row-vector $\left[\widetilde{\mathcal{P}}_l\right]$ be given by* (7)*. Then the following function $e^{ix_0 \cdot x}\left[\widetilde{\mathcal{P}}_l(x)\right] v$, $v \in \mathbb{C}^{\widetilde{d}(l)}$, is non-zero and belongs to* $\ker P(-iD)$ *if and only if $P(x_0) = 0$ and $v \in V_l$.*

Proof of Theorem 3.1. As it has been said, see Theorem 2.4, a necessary and sufficient condition for the matrix $\widetilde{\mathbf{D}}_l P(x_0)$ to be singular is that the point x_0 be a root of the polynomial P.

For some $v \in \mathbb{C}^{\widetilde{d}(l)}$, consider a function

$$G(x) := P(-iD)\left(e^{ix_0 \cdot x}\left[\widetilde{\mathcal{P}}_l(x)\right] v\right).$$

Taking the Fourier transform of the previous function, we obtain

$$\hat{G}(\xi) := P(\xi)\left[\widetilde{\mathcal{D}}_l \delta(\xi - x_0)\right] v, \qquad \xi \in \mathbb{C}^d. \tag{20}$$

The adjoint operator (set of operators) $\widetilde{\mathcal{D}}_l^*$ satisfies a property

$$\widetilde{\mathcal{D}}_l^* = \overline{\widetilde{\mathcal{D}}_l} \tag{21}$$

(the adjunction, like the complex conjugation, is distributive over the comma, see Remark 1.3).

By Definition 1.2, Theorem 2.1, and property (21); for any test function $\phi \in S(\mathbb{C}^d)$, the functional $T_{\hat{G}}(\phi)$ (where $\hat{G}$ is distribution (20)) is of the form

$$\begin{aligned} T_{\hat{G}}(\phi) &= \overline{\left\langle P(\cdot)\left[\widetilde{\mathcal{D}}_l \delta(\cdot - x_0)\right] v \middle| \phi\right\rangle} = \overline{\left\langle \delta(\cdot - x_0) \middle| \left[\overline{\widetilde{\mathcal{D}}_l}\left(\overline{P}\phi\right)\right] \overline{v}\right\rangle} \\ &= \left[\widetilde{\mathcal{D}}_l\left(P(x_0)\overline{\phi(x_0)}\right)\right] v \overset{\text{by (13)}}{=} \left[\widetilde{\mathcal{D}}_l \overline{\phi(x_0)}\right]\left[\widetilde{\mathbf{D}}_l P(x_0)\right] v. \end{aligned} \tag{22}$$

Using the expression in the right-hand side of (22), the proof of the theorem is trivial. □

Remark 3.1. Theorem 3.1 can be proved for other (linear and linear-conjugate, see for example [11]) functionals like (4).

Below present a corollary of the previous theorem.

Corollary 3.2. *Under the conditions of Theorem 3.1, we see that the following spaces are equivalent:*

$$e^{ix_0\cdot x}\Pi_{\leq l}\cap \ker P(-iD);$$
$$e^{ix_0\cdot x}\left\{\left[\widetilde{\mathcal{P}}_l(x)\right]v : v\in V_l\right\}.$$

Now we state a theorem that is the direct consequence of Theorem 2.6.

Theorem 3.3. *Let $l\in\mathbb{Z}_{\geq 0}$. Let $P\in\Pi$, let x_0 be a root of P. Let $D^\alpha P(x_0)$, $\alpha\in\mathbb{Z}^d_{\geq 0}$, be a non-zero derivative of the least order. Then we have*

$$\dim\left(e^{ix_0\cdot x}\Pi_{\leq l}\cap\ker P(-iD)\right)=\begin{cases}\widetilde{d}(l)-\widetilde{d}(l-|\alpha|) & \text{if } l\geq|\alpha|>0;\\ \widetilde{d}(l) & \text{if } l<|\alpha|.\end{cases}$$

Moreover, if $l<|\alpha|$; then

$$e^{ix_0\cdot x}\Pi_{\leq l}\cap\ker P(-iD)=e^{ix_0\cdot x}\Pi_{\leq l}.$$

Below we state a corollary of Theorem 2.8.

Corollary 3.4. *Under the conditions of Theorem 3.1, we see that if x_0 is a root of the polynomial P, then the null-space of the operator $P(-iD)$ contains polynomials (multiplied by the exponential $e^{ix_0\cdot x}$) up to an arbitrary large total degree.*

Remark 3.2. Corollary 3.4 generalizes a fundamental property, see [1], of polynomial solutions to a single PDE with constant coefficients.

3.2. System of homogeneous PDE's

For a system of PDE's

$$\begin{cases}P_1(-iD)\cdot=0,\\ \vdots\\ P_n(-iD)\cdot=0,\end{cases}\tag{23}$$

where $P_1,P_2,\ldots,P_n$ are algebraic polynomials; we have the following theorem.

Theorem 3.5. *Let $P_m\in\Pi$, $m=1,2,\ldots,n$; let $x_0\in\mathbb{C}^d$ be a solution to the following system of algebraic equations*

$$\begin{cases}P_1(x)=0,\\ \vdots\\ P_n(x)=0.\end{cases}\tag{24}$$

Let $l \in \mathbb{Z}_{\geq 0}$ and the matrix $\widetilde{\mathbf{D}}_l$ be given by (11)*. The expression $e^{ix_0 \cdot x} \left[\widetilde{\mathcal{P}}_l\right] v$, where $v \in \mathbb{C}^{\widetilde{d}(l)}$, is a non-zero solution to system* (23) *iff the vector v belongs to the null-space of block matrix*

$$\widetilde{\mathbf{D}}_l P(x_0) := \begin{bmatrix} \widetilde{\mathbf{D}}_l P_1(x_0) \\ \vdots \\ \widetilde{\mathbf{D}}_l P_n(x_0) \end{bmatrix}, \tag{25}$$

where P is the column vector $P := \begin{bmatrix} P_1 & P_2 & \cdots & P_n \end{bmatrix}^T$.

Definition 3.1. Under the conditions of Theorem 3.5, we can define the following polynomial space

$$\mathcal{V}_l := \left\{ \left[\widetilde{\mathcal{P}}_l\right] v : v \in \ker \widetilde{\mathbf{D}}_l P(x_0) \right\}, \tag{26}$$

where $\widetilde{\mathbf{D}}_l P(x_0)$ is block matrix (25).

Below we state an almost obvious theorem about non-zero solutions to system of PDE's (23).

Theorem 3.6. *Let $P_m \in \Pi$, $m = 1, 2, \dots, n$. Let $x_0 \in \mathbb{C}^d$, $l \in \mathbb{Z}_{\geq 0}$, and the matrix $\widetilde{\mathbf{D}}_l P(x_0)$, $P := \begin{bmatrix} P_1 & P_2 & \cdots & P_n \end{bmatrix}^T$, be given by* (25)*. Let $D^{\alpha_m} P_m(x_0)$, $\alpha_m \in \mathbb{Z}^d_{\geq 0}$, $m = 1, \dots, n$, be the non-zero derivatives of the least orders. Let lth column (block) vector be in the form*

$$\begin{bmatrix} \mathbf{D}_l^{l-|\alpha_1|} P_1(x_0) \\ \mathbf{D}_l^{l-|\alpha_2|} P_2(x_0) \\ \cdots \\ \mathbf{D}_l^{l-|\alpha_n|} P_n(x_0) \end{bmatrix}, \tag{27}$$

where the submatrices $\mathbf{D}_l^{l-|\alpha_m|}$, $m = 1, 2, \dots, n$, are defined by (10)*. (If, for some $m \in \{1, \dots, n\}$, $l - |\alpha_m| < 0$; then the corresponding submatrix $\mathbf{D}_l^{l-|\alpha_m|} P_m(x_0)$ is a zero matrix, see Remark 2.1.) Suppose the polynomial space $\mathcal{V}_l$ corresponds to block matrix* (25)*, see Definition 3.1. Then $\mathcal{V}_l \cap \Pi_l \neq \emptyset$ iff matrix* (27) *is not a full-rank matrix.*

Now we can state several corollaries of Theorem 3.6.

Corollary 3.7. *Suppose that, for some $l \in \mathbb{N}$, matrix* (27) *is not full-rank; then, for all $l' \in \mathbb{Z}_{\geq 0}$, $0 \leq l' < l$, we have $\mathcal{V}_{l'} \cap \Pi_{l'} \neq \emptyset$, where $\mathcal{V}_{l'} := \ker \widetilde{\mathbf{D}}_{l'} P(x_0)$, and $\widetilde{\mathbf{D}}_{l'}$ is an analog of matrix* (25)*.*

Corollary 3.8. *If $l < \min\{|\alpha_1|, \dots, |\alpha_n|\}$, then $\mathcal{V}_l = \Pi_{\leq l}$.*

Corollary 3.9. *For the block matrix*

$$\begin{bmatrix} \mathbf{D}_0^0 P_1(x_0) \\ \mathbf{D}_0^0 P_2(x_0) \\ \cdots \\ \mathbf{D}_0^0 P_n(x_0) \end{bmatrix}$$

to be no full-rank matrix it is necessary and sufficient to have

$$P_1(x_0) = \cdots = P_n(x_0) = 0. \tag{28}$$

Remark 3.3. Thus system (23) has a nonzero polynomial (multiplied by $e^{ix_0 \cdot x}$) solution iff conditions (28) are valid.

Theorem 3.10. *Under the conditions of Theorem 3.5, we have*

$$\mathcal{V}_{l-1} = \Pi_{\leq l-1} \cap \mathcal{V}_l, \tag{29}$$

where the polynomial spaces $\mathcal{V}_{l-1}, \mathcal{V}_l$, $l \in \mathbb{N}$, are defined by (26).

The proof of Theorem 3.10 is given in B.

Finally we state three remarks.

Remark 3.4.

(1) Theorem 3.10 is valid also for single PDE.
(2) Certainly, inclusion (29) of polynomial spaces reflects a fundamental property of differentiation to commutate with translation.
(3) Note also that if $\mathcal{V}_l \subseteq \Pi_{\leq l-1}$ (it is possible only for a system of two and more PDE's), then $\mathcal{V}_{l-1} = \mathcal{V}_l$.

3.3. PDE with a polynomial right-hand side

In the previous subsections, PDE's have zero right-hand sides. Nevertheless the matrix approach allows generalizing PDE to polynomial (multiplied by an exponential) right-hand side.

Theorem 3.11. *Let polynomials $P, F \in \Pi$, $x_0 \in \mathbb{C}^d$ be a* root *of P, $\alpha \in \mathbb{Z}^d_{\geq 0}$ be a multi-index that defines the least order derivative such that $D^\alpha P(x_0) \neq 0$. Let the polynomial F be defined as follows: $F(x) := \left[\widetilde{\mathcal{P}}_{\deg F}(x)\right] w$, $w \in \mathbb{C}^{\widetilde{d}(\deg F)}$. Let $l \in \mathbb{Z}_{\geq 0}$, $l \geq \deg F + |\alpha|$, and the matrix $\widetilde{\mathbf{D}}_l$ be given by* (11)*. Let $v \in \mathbb{C}^{\widetilde{d}(l)}$ be a column vector and $p := \left[\widetilde{\mathcal{P}}_l\right] v$ be the corresponding polynomial. Then the polynomial p is a solution to PDE*

$$P(-iD)\left(e^{ix_0 \cdot x} \cdot\right) = e^{ix_0 \cdot x} F(x) \tag{30}$$

iff the vector v is a solution to linear algebraic equation

$$\left[\widetilde{\mathbf{D}}_l P(x_0)\right] v = \begin{bmatrix} w \\ \left.\begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix}\right\} \widetilde{d}(l) - \widetilde{d}(\deg F) \end{bmatrix}. \tag{31}$$

Remark 3.5. Under the conditions of Theorem 3.11, we can state three remarks:

(1) $\dim \{p \in \Pi_{\leq l} : P(-iD)\left(e^{ix_0 \cdot x} p(x)\right) = e^{ix_0 \cdot x} F(x)\} = \widetilde{d}(l) - \widetilde{d}(l - |\alpha|)$;
(2) for any root $x_0 \in \mathbb{C}^d$ of P, number $l \in \mathbb{Z}_{\geq 0}$, and polynomial $F \in \Pi$ such that $\deg F \leq l - |\alpha|$; algebraic system (31) is consistent;

(3) for any root $x_0 \in \mathbb{C}^d$ of P and any polynomial $F \in \Pi$, there exists a polynomial $p \in \Pi$' of an arbitrary large degree that p satisfies PDE (30).

Corollary 3.12. *Under the conditions of the previous theorem, and supposing* $P(x_0) \neq 0$*; we see that algebraic system* (31) *is consistent and has* one *solution, i. e., polynomial solution to PDE* (30) *is defined uniquely and does not depend on the choice of* l.

The proof of the previous results is left to the reader. Note only that the theorem and remarks are based on Theorem 2.6, Theorem 3.1, and the classical Rouché-Capelli theorem.

3.4. Affine invariance

In this short subsection, we concern affine invariance of polynomial solutions to PDE's, in particular, from the matrix point of view.

Recall that a (polynomial) space $\mathcal{V} \subset \Pi$ is *affinely invariant*, i. e., shift- and scale-invariant; if, for all $p \in \mathcal{V}$, $p(ax+b) \in \mathcal{V}$ for $a \in \mathbb{C}$, $b \in \mathbb{C}^d$. Also recall that a polynomial space $\mathcal{V}$ is scale-invariant iff $\mathcal{V}$ *stratifies:*

$$\mathcal{V} = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \mathcal{V} \cap \Pi_k. \tag{32}$$

Stratified form (32) of the polynomial space $\mathcal{V}_l$, $l \in \mathbb{N}$, implies the following conditions on the null-space V_l (see Definitions 3.1 and 3.2)

$$V_l = \bigoplus_{k=0}^{l} \mathfrak{P}_k V_l = \bigoplus_{k=0}^{l} V_l \cap {}^k\mathbb{C}^{\widetilde{d}(l)},$$

cf. (16), and vice verse. Note, for affine invariance case, we have

$$\mathfrak{P}_k V_l = V_l \cap {}^k\mathbb{C}^{\widetilde{d}(l)}.$$

Thus the column matrix $[V_l]$ of basis vectors can be presented in a block diagonal form.

The matrix $\widetilde{\mathbf{D}}_l P(x_0)$, see (19), for the affine invariance of a polynomial space, will be the object of another paper.

4. Examples. From the practical point of view, define a system of algebraic polynomials that induces PDE's, fixing some number $l \in \mathbb{Z}_{\geq 0}$ and a point $x_0 \in \mathbb{C}^c\widetilde{d}(l)$, define the null-space (using some computer symbolic algebra system) $\ker \widetilde{\mathbf{D}}_l P(x_0)$ as a matrix $[V_l]$ of column basis vectors; then $e^{ix_0 \cdot x} \left[\widetilde{\mathcal{P}}_l(x)\right] [V_l]$ is the row-vector of polynomials (multiplied by the exponential $e^{ix_0 \cdot x}$)) that this vector constitutes a basis of space $[\mathcal{V}_l]$, see (26).

4.1. Homogeneous equations

First we consider three examples, presented in the paper [4], of polynomial solution to PDE's.

Example 1. The first example is 2D Laplace operator

$$L_1 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad x, y \in \mathbb{R}. \tag{33}$$

So the polynomial that induces the operator is

$$P_1(\zeta, \eta) := -\zeta^2 - \eta^2, \qquad \zeta, \eta \in \mathbb{C}. \tag{34}$$

The 10×10 matrix $\widetilde{\mathbf{D}}_3 P_1(0,0)$ is of the form

$$\widetilde{\mathbf{D}}_3 P_1(0,0) := \left[\begin{array}{c|cc|ccc|cccc} 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 6 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{35}$$

Now, selecting the appropriate set of basis vectors

$$\left[\ker \widetilde{\mathbf{D}}_3 P_1(0,0)\right] := \left[\begin{array}{c|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

we obtain the well-known row-vector of basis polynomials

$$\left[\widetilde{\mathcal{P}}_3(x,y)\right]\left[\ker \widetilde{\mathbf{D}}_3 P_1(0,0)\right] = \begin{bmatrix} 1 & x & y & xy & y^2 - x^2 & 3xy^2 - x^3 & y^3 - 3x^2 y \end{bmatrix}. \tag{36}$$

Example 2. The polynomial $P_2(\zeta, \eta) := -\zeta^2 - i\eta$ that induces the operator of this example

$$L_2 := \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y}$$

is not homogeneous. The matrix $\widetilde{\mathbf{D}}_3 P_2(0,0)$ is of the form

$$\widetilde{\mathbf{D}}_3 P_2(0,0) := \left[\begin{array}{c|cc|ccc|cccc} 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

and, using some matrix $\left[\ker \widetilde{\mathbf{D}}_3 P_2(0,0)\right]$, a basis of $\ker L_2(-iD) \cap \Pi_{\leq 3}$ can be in the form

$$\left\{1, x, x^2 + 2y, x^3 + 6xy\right\}.$$

Example 3. The third example taken from the paper [4] is interesting from two points of view. Namely we have a system of two operators: an elliptic (the Laplace operator)

$$L_3 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and hyperbolic

$$L_4 := \frac{\partial}{\partial x}\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\frac{\partial}{\partial z} + \frac{\partial}{\partial y}\frac{\partial}{\partial z}.$$

And it is a 3D example.

Since, in [4], the third degree polynomials are considered only; therefore, we use the last (block) columns of matrices $\widetilde{\mathbf{D}}_3 P_3(0,0)$, $\widetilde{\mathbf{D}}_3 P_4(0,0)$ (it is possible only in an affine-invariant case), where P_3, P_4 are the algebraic polynomials that corresponds to the operators L_3, L_4, respectively.

$$\left[\begin{array}{c} \mathbf{D}_3 P_3(0,0) \\ \hline \mathbf{D}_3 P_4(0,0) \end{array}\right] = \left[\begin{array}{c} \mathbf{D}_3^0 P_3(0,0) \\ \vdots \\ \mathbf{D}_3^3 P_3(0,0) \\ \hline \mathbf{D}_3^0 P_4(0,0) \\ \vdots \\ \mathbf{D}_3^3 P_4(0,0) \end{array}\right].$$

Since other blocks of the previous matrix are zero, we have

$$\ker\left[\frac{\mathbf{D}_3P_3(0,0)}{\mathbf{D}_3P_4(0,0)}\right]=\ker\left[\begin{array}{cccccccccc} 6&0&2&0&0&0&0&2&0&0\\ 0&2&0&6&0&0&0&0&2&0\\ 0&0&0&0&2&0&2&0&0&6\\ \hline 0&2&0&0&2&1&0&0&0&0\\ 0&0&2&0&0&1&2&0&0&0\\ 0&0&0&0&0&1&0&2&2&0 \end{array}\right].$$

Using a column-matrix $\left[\ker\left[\frac{\mathbf{D}_3P_3(0,0)}{\mathbf{D}_3P_4(0,0)}\right]\right]$, we get a basis of the space $\ker L_3\cap\ker L_4\cap\Pi_3$:

$$\begin{aligned}\{3x^2y-3x^2z-y^3+z^3, -x^3+3x^2y+3xy^2-6xyz-2y^3+3yz^2\ ,\\ -2x^3+3x^2y+3xy^2-6xyz+3xz^2-y^3,\\ x^3+3x^2y-3x^2z-3xy^2-y^3+3y^2z\}\,.\end{aligned}$$

Note that, in the paper [4], another basis is presented. (It depends on basis vectors of the null-space.) But it is not hard to see that our own and Pedersen's, see [4], basis are bases of the same space.

Secondly we present an example, where another (not the origin) root of polynomial is used.

Example 4. This example is taken from the paper [9]. Since the symbol $P_1(\zeta,\eta)$, $\zeta,\eta\in\hat{\mathbb{C}}$, of 2D Laplace operator (33) vanishes on a 2D manifold; we can take another root of P_1 than the origin. Here we take, as an example, a root $(1,i)$; and we obtain the following matrix

$$\widetilde{\mathbf{D}}_3P_1(1,i):=\left[\begin{array}{c|cc|ccc|cccc} 0&-2i&2&-2&0&-2&0&0&0&0\\ 0&0&0&-4i&2&0&-6&0&-2&0\\ 0&0&0&0&-2i&4&0&-2&0&-6\\ 0&0&0&0&0&0&-6i&2&0&0\\ 0&0&0&0&0&0&0&-4i&4&0\\ 0&0&0&0&0&0&0&0&-2i&6\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{array}\right].$$

So a subspace of Laplace's operator null-space is of the form

$$\begin{aligned}&e^{ix-y}\Pi_{\le 3}\cap\ker L_1\\ &\quad=e^{ix-y}\operatorname{span}\left\{1,x+iy,x^2+2ixy-y^2,x^3+3ix^2y-3xy^2-iy^3\right\}.\end{aligned}\qquad(37)$$

Note that the real or imaginary parts of the polynomials in (37) (cf. (36) and (37)) multiplied by an exponential, do not become the solutions to the Laplace operator.

Finally we consider an operator that the operator symbol does not vanish at the origin, thus any pure polynomial must be multiplied by an exponential to be solution to PDE.

Example 5. Consider the Helmholtz operator

$$L_5 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - I\,.$$

The corresponding symbol is

$$P_5(\zeta, \eta) := -\zeta^2 - \eta^2 - 1, \qquad \zeta, \eta \in \mathbb{C}. \tag{38}$$

P_5 vanishes, in particular, at a point $(i, 0)$; and we consider the following matrix

$$\widetilde{\mathbf{D}}_3 P_5(i,0) := \left[\begin{array}{c|cc|ccc|cccc} 0 & -2 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -4 & 0 & 0 & 6 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 2 & 0 & 6 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

And we have

$$e^{-x}\Pi_{\leq 3} \cap \ker L_5 = e^{-x} \operatorname{span}\left\{1, y, x + y^2, 3xy + y^3\right\}.$$

4.2. PDE with polynomial right-hand side

Finally we discuss PDE's with polynomial (multiplied, in general, by an exponential) right-hand sides.

Example 6. Here we present the Poisson equation in the form

$$L_1\left(e^{ix-y}\cdot\right) = e^{ix-y}\left(-2xy + 3x + y^2 + 2\right), \tag{39}$$

where L_1 is 2D Laplace's operator (33) and the point $(1, i)$, which defines the exponential e^{ix-y}, is a root of the corresponding symbol P_1, see (34). The polynomial in the right-hand side of PDE (39) can be presented as

$$F(x,y) := -2xy + 3x + y^2 + 2 = \left[\widetilde{\mathcal{P}}_2(x,y)\right] w, \quad \text{where}$$
$$w := \begin{bmatrix} 2 & 3 & 0 & 0 & -2 & 1 \end{bmatrix}^T.$$

Since $\deg F = 2$ and $|\alpha| = 1$, where $\alpha \in \mathbb{Z}^d_{\geq 0}$ is minimal degree such that $D^\alpha P_1(1,i) \neq 0$; we shall use $l = 3$.

The corresponding linear algebraic equation (31) is:

$$\left[\begin{array}{c|cc|ccc|cccc} 0 & 2i & -2 & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 4i & -2 & 0 & 6 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2i & -4 & 0 & 2 & 0 & 6 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 6i & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4i & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2i & -6 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] v = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{where } v \in \mathbb{C}^{10}. \tag{40}$$

By Item (1) of Remark 3.5, the dimension of polynomial space to solve PDE (39) (and dimension on linear space to solve (40)) is $\widetilde{d}(3) - \widetilde{d}(2) = d(3) = 4$. Since linear algebraic system (40) is undetermined and consistent; using the standard technics, we define the solution to (40) (column-vector v) and obtain a polynomial solution to PDE (39)

$$\begin{aligned} &\left(\frac{1}{6} + \frac{i}{6}\right)(-1 + (3+3i)v_4)x^3 + v_3\left(-x^2 - 2ixy + y^2\right) \\ &\quad + \left(-3v_4 + \left(\frac{1}{2} - \frac{i}{2}\right)\right)x^2y - \frac{1}{2}i(6v_4 - 1)xy^2 + v_2(x+iy) + v_4y^3 + v_1 \\ &\quad + \left(\frac{1}{4} + \frac{i}{4}\right)x^2 + \left(\frac{1}{2} + \frac{i}{2}\right)xy + \left(\frac{5}{4} + \frac{i}{4}\right)y, \qquad v_1, v_2, v_3, v_4 \in \mathbb{C}. \end{aligned}$$

Example 7. In this example, we consider Poisson's equation. However we use a point $(1,1)$ that is not root of P_1. We solve the following PDE:

$$L_1\left(e^{ix+iy}\cdot\right) = e^{ix+iy}(3 + x - 2y), \tag{41}$$

where L_1 is Laplace operator (33). The value $l = 1$ will suffice and the linear system is very simple

$$\left[\begin{array}{c|cc} -2 & 2i & 2i \\ \hline 0 & -2 & 0 \\ 0 & 0 & -2 \end{array}\right] v = \left[\begin{array}{c} 3 \\ \hline 1 \\ -2 \end{array}\right]$$

and the unique polynomial solution to PDE (39) is of the form

$$-\frac{x}{2} + y - \left(\frac{3}{2} - \frac{i}{2}\right).$$

Conclusion. In this paper, a matrix method has been developed for the constructive determination of polynomial solutions of linear PDEs with constant coefficients. Here we will present some plans for the near future. It is of particular interest to generalize the method to the following cases:

- matrix methods of polynomial solutions to PDE('s) with *polynomial coefficients*;

- matrix methods of investigation of polynomial solutions to *systems* of constant coefficient PDE's;
- matrix methods of investigation of *affine invariance/no invariance* of polynomial solution spaces to constant coefficient PDE('s).

A. Proof of Theorem 2.5 First consider the matrix $\mathbf{D}_m^0 f(x_0)$, $m = 1, \dots, l$. We see that $\mathbf{D}_m^0 f(x_0)$ is a row vector $\left[(-i)^m D^{{}^q\alpha} f(x_0)\right]$ $(1 \le q \le d(m))$. Consequently the matrix $\mathbf{D}_m^0 f(x_0)$ has non-zero rank iff there exists at least one multi-index ${}^q\alpha \in \mathcal{A}_m$ such that $D^{{}^q\alpha} f(x_0) \ne 0$.

Secondly consider the matrices $\mathbf{D}_m^{m'} f(x_0)$, $m' = 1, \dots, m-1$, $m = 1, \dots, l$. Note that $\mathbf{D}_m^{m'} f(x_0)$ is a $d(m') \times d(m)$ matrix. By ${}^{qr}\gamma$, $1 \le q \le d(m')$, $1 \le r \le d(m)$, denote the difference ${}^q\alpha - {}^r\beta$, where $\left({}^1\alpha, \dots, {}^{d(m)}\alpha\right) = \mathcal{A}_m$ and $\left({}^1\beta, \dots, {}^{d(m')}\beta\right) = \mathcal{A}_{m'}$. Define auxiliary $d(m') \times d(m)$ matrices $\mathbf{G}_m^{m'}$, $m' = 1, \dots, m-1$, $m = 1, \dots, l$, as follows

$$\left[\mathbf{G}_m^{m'}\right]_{qr,\ 1\le q\le d(m'),\, 1\le r\le d(m)} = {}^{qr}\gamma. \tag{42}$$

We suppose that some entries of the matrix $\mathbf{G}_m^{m'}$ do not belong to $\mathbb{Z}^d_{\ge 0}$, i. e., a tuple ${}^{qr}\gamma$ can contain negative components.

Lemma A.1. *Any multi-index $\gamma \in \mathcal{A}_{m-m'}$ appears once in each row and no more than once in each column of the matrix $\mathbf{G}_m^{m'}$. However not every column of $\mathbf{G}_m^{m'}$ contains the multi-index γ.*

Proof of Lemma A.1. For any multi-index $\gamma \in \mathcal{A}_{m-m'}$ and any row q, $q = 1, \dots, d(m')$, of the matrix $\mathbf{G}_m^{m'}$, we can always define a d-tuple $\alpha \in \mathbb{Z}^d$ as $\alpha := \gamma + {}^q\beta$, where ${}^q\beta \in \mathcal{A}_{m'}$. Since ${}^q\beta, \gamma \in \mathbb{Z}^d_{\ge 0}$ and $|{}^q\beta| = m'$, $|\gamma| = m - m'$; therefore, $\alpha \in \mathbb{Z}^d_{\ge 0}$, $|\alpha| = m$. Consequently there exists a unique (column) number $r \in \{1, \dots, d(m)\}$ such that $\alpha = {}^r\alpha \in \mathcal{A}_m$.

Fix a column number $r \in \{1, \dots, d(m)\}$. Consider some $\gamma \in \mathcal{A}_{m-m'}$ and define a d-tuple β as $\beta := {}^r\alpha - \gamma$, where ${}^r\alpha \in \mathcal{A}_m$. If $\gamma \le {}^r\alpha$, then $\beta = {}^q\beta \in \mathcal{A}_m$ for a unique (row) number $q \in \{1, \dots, d(m')\}$; else the column r does not contain the multi-index γ. □

Lemma A.2. *Let $\left({}^1\beta, \dots, {}^{d(m')}\beta\right) = \mathcal{A}_{m'}$, $\left({}^1\gamma, \dots, {}^{d(m-m')}\gamma\right) = \mathcal{A}_{m-m'}$, $m > m'$. Consider an entry ${}^{qr}\gamma$ of the matrix $\mathbf{G}_m^{m'}$ and suppose that ${}^{qr}\gamma$ is equal to a multi-index ${}^j\gamma \in \mathcal{A}_{m-m'}$, $j \in \{1, \dots, d(m-m')\}$; then any entry of the rth column of $\mathbf{G}_m^{m'}$ below than ${}^{qr}\gamma$, i. e., the entry ${}^{q'r}\gamma$, $q' \in \{q+1, \dots, d(m')\}$, belongs either to the set of the multi-indices $\{{}^1\gamma, \dots, {}^{j-1}\gamma\}$ or does not belong to $\mathbb{Z}^d_{\ge 0}$.*

Proof of Lemma A.2. Since the multi-indices ${}^q\beta$, $q = 1, \dots, d(m')$, are lexicographically ordered, i. e., ${}^1\beta <_{\text{lex}} {}^2\beta <_{\text{lex}} \dots <_{\text{lex}} {}^{d(m')}\beta$; using (42), we obtain the order

$${}^{1r}\gamma >_{\text{lex}} {}^{2r}\gamma >_{\text{lex}} \dots >_{\text{lex}} {}^{d(m'),r}\gamma. \tag{43}$$

Consider any ${}^{q'r}\gamma$, $q' \in \{q+1, \dots, d(m')\}$, assume ${}^{q'r}\gamma \notin \{{}^1\gamma, \dots, {}^{j-1}\gamma\}$ and ${}^{q'r}\gamma \in \mathbb{Z}^d_{\ge 0}$. Then there exists a unique number $j' \in \{j, \dots, d(m-m)\}$ such that ${}^{j'}\gamma = {}^{q'r}\gamma$. Since

$j' \geq j$, we have ${}^{j'}\gamma = {}^{q'r}\gamma \geq_{\rm lex} {}^{j}\gamma = {}^{qr}\gamma$. By (43), we get $q' \leq q$. This contradiction concludes the proof. □

Finally let us prove the theorem.

Proof of Theorem 2.5. Obviously that at least one non-zero derivative $D^{\gamma}f(x_0)$, $|\gamma| = m - m'$, is necessary for full rank of the matrix $\mathbf{D}_m^{m'}f(x_0)$.

Sufficiency. For some ${}^{j}\gamma \in \mathcal{A}_{m-m'}$, $j \in \{1, \ldots, d(m-m')\}$, suppose that the derivative $D^{{}^{j}\gamma}f(x_0) \neq 0$. By Lemma A.1, from $\mathbf{D}_m^{m'}f(x_0)$, we can take a $d(m') \times d(m')$ submatrix with the non-zero main diagonal $\left(D^{{}^{j}\gamma}f(x_0), \ldots, D^{{}^{j}\gamma}f(x_0)\right)$ and we denote this submatrix by $\mathbf{S}_j$.

Suppose, for ${}^{1}\gamma \in \mathcal{A}_{m-m'}$, $D^{{}^{1}\gamma}f(x_0) \neq 0$; then, by Lemma A.2 and property (1), all the entries below the main diagonal of the submatrix $\mathbf{S}_1$ vanish. So the matrix $\mathbf{S}_1$ is not singular, consequently $\operatorname{rank} \mathbf{D}_m^{m'}f(x_0) = d(m')$.

Otherwise, $D^{{}^{1}\gamma}f(x_0) = 0$. Suppose there exists a multi-index ${}^{j}\gamma \in \mathcal{A}_{m-m'}$, $j \in \{2, \ldots, d(m-m')\}$, such that $D^{{}^{j}\gamma}f(x_0) \neq 0$ and all the derivatives $D^{{}^{j'}\gamma}f(x_0)$, $j' = 1, \ldots, j-1$, vanish. Since all the entries below the main diagonal of the matrix $\mathbf{S}_j$ vanish, it follows that $\det \mathbf{S}_j \neq 0$. This concludes the proof of the sufficiency. □

B. Proof of Theorem 3.10 We must restructure the matrices $\mathbf{D}_k f(x_0)$ (see (9)), where $f := \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$, $k = 0, \ldots l$; and introduce a notation.

Let $k, l \in \mathbb{Z}_{\geq 0}$, $k \leq l$. Define the $n\widetilde{d}(l) \times d(k)$ matrix $\breve{\mathbf{D}}_k f(x_0)$, as follows

$$\breve{\mathbf{D}}_k f(x_0) := \begin{bmatrix} \breve{\mathbf{D}}_k^0 f(x_0) \\ \breve{\mathbf{D}}_k^1 f(x_0) \\ \vdots \\ \breve{\mathbf{D}}_k^k f(x_0) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \text{where} \quad \breve{\mathbf{D}}_k^r f(x_0) := \begin{bmatrix} \mathbf{D}_k^r f_1(x_0) \\ \mathbf{D}_k^r f_2(x_0) \\ \vdots \\ \mathbf{D}_k^r f_n(x_0) \end{bmatrix} \tag{44}$$

and $\mathbf{D}_k^r$, $r = 0, \ldots k$, $k = 0, \ldots l$, is given by (10).

And formulate an analog of matrix (19) (see also (11))

$$\widetilde{\breve{\mathbf{D}}}_l f(x_0) := \begin{bmatrix}\breve{\mathbf{D}}_0 f(x_0) & \breve{\mathbf{D}}_1 f(x_0) & \dots & \breve{\mathbf{D}}_l f(x_0)\end{bmatrix}$$
$$= \begin{bmatrix} \breve{\mathbf{D}}_0^0 f(x_0) & \breve{\mathbf{D}}_1^0 f(x_0) & \dots & \breve{\mathbf{D}}_{l-1}^0 f(x_0) & \breve{\mathbf{D}}_l^0 f(x_0) \\ 0 & \breve{\mathbf{D}}_1^1 f(x_0) & \dots & \breve{\mathbf{D}}_{l-1}^1 f(x_0) & \breve{\mathbf{D}}_l^1 f(x_0) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \breve{\mathbf{D}}_{l-1}^{l-1} f(x_0) & \breve{\mathbf{D}}_l^{l-1} f(x_0) \\ 0 & 0 & \dots & 0 & \breve{\mathbf{D}}_l^l f(x_0) \end{bmatrix}. \tag{45}$$

Remark A.1. Obviously, the matrices $\widetilde{\mathbf{D}}_l f(x_0)$, $\widetilde{\breve{\mathbf{D}}}_l f(x_0)$ have the same null-space.

Proof of Theorem 3.10. Suppose a polynomial $p \in \mathcal{V}_{l-1}$. Then there exists a vector $v \in \ker \widetilde{\breve{\mathbf{D}}}_{l-1} f(x_0)$ such that $p = \left[\widetilde{\mathcal{P}}_{l-1}\right] v$. Obviously,

$$p \in \Pi_{\le l-1}. \tag{46}$$

Matrix (45) can be presented as a block matrix

$$\widetilde{\breve{\mathbf{D}}}_l P(x_0) := \left[\begin{array}{ccc|c} & & & \breve{\mathbf{D}}_l^0 P(x_0) \\ & \widetilde{\breve{\mathbf{D}}}_{l-1} P(x_0) & & \vdots \\ & & & \breve{\mathbf{D}}_l^{l-1} P(x_0) \\ \hline 0 & \dots & 0 & \breve{\mathbf{D}}_l^l P(x_0) \end{array}\right], \tag{47}$$

where the submatrices $\breve{\mathbf{D}}_l^0, \dots, \breve{\mathbf{D}}_l^l$ are given by (44). Introduce an auxiliary column vector as follows

$$v^\sharp := \begin{bmatrix} v \\ d(l) \left\{ \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \right. \end{bmatrix} \tag{48}$$

Then, using block form (47) of matrix $\widetilde{\breve{\mathbf{D}}}_l P(x_0)$, we get $v^\sharp \in \ker \widetilde{\mathbf{D}}_l P(x_0)$. Since $p = \left[\widetilde{\mathcal{P}}_l\right] v^\sharp$, it follows that $p \in \mathcal{V}_l$; and, by (46), we obtain $p \in \Pi_{\le l-1} \cap \mathcal{V}_l$. Thus we have $\mathcal{V}_{l-1} \subseteq \Pi_{\le l-1} \cap \mathcal{V}_l$.

Contrary. Suppose a polynomial $p \in \Pi_{\le l-1} \cap \mathcal{V}_l$. Since $p \in \Pi_{\le l-1}$, it follows that the polynomial can be presented as follows $p = \left[\widetilde{\mathcal{P}}_l\right] v^\sharp$, where $v^\sharp$ is given by (48). Since $v^\sharp \in \ker \widetilde{\mathbf{D}}_l P(x_0)$ and, arguing as above; $v \in \ker \widetilde{\mathbf{D}}_{l-1} P(x_0)$. Thus $p \in \mathcal{V}_{l-1}$; i. e., we have $\mathcal{V}_{l-1} \supseteq \Pi_{\le l-1} \cap \mathcal{V}_l$.

This concludes the proof. □

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